Optimization I	**Summer 2011** **Homework 4**

Duality

This assignment is due on **May 23 in lecture**. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be *written individually* without consulting someone else's solutions or any other source like the web.

1. Construct the dual of

$$\begin{array}{llllllllll}
\min & x_1 & - & x_2 & & & & & & \\
\text{s.t.} & 2x_1 & + & 3x_2 & - & x_3 & + & x_4 & \leq & 0 \\
& 3x_1 & + & x_2 & + & 4x_3 & - & 2x_4 & \geq & 3 \\
& -x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \\
& x_2 & \geq 0, & x_3 & \geq 0. & & & & &
\end{array}$$

2. Consider the shortest path problem we discussed in lecture: We are given a directed graph $G = (V, E)$. Let $|V| = m, |E| = n$. We assume the nodes are labelled $v_1, \ldots, v_m$, and the edges are labelled $e_1, \ldots, e_n$. There is a cost/length $c_j \geq 0$ associated with each edge $e_j \in E$.

   Your friend has implemented Dijkstra's algorithm and his program outputs the shortest path from v_1 to v_m, and it outputs a value $d(i)$ for each i, which is equal to the length of the shortest path from v_i to v_m.

   However, your friend is not a very good programmer, and you are worried that the output may not be correct. Explain how you can verify in $O(|E|)$ time whether the path output by your friend's program is indeed the shortest path from v_1 to v_m.

3. Consider the following linear program

$$\begin{array}{lllll}
\min & -10x_1 & -12x_2 & -12x_3 & \\
\text{s.t.} & x_1 & +2x_2 & +2x_3 & \leq 20 \\
& 2x_1 & +x_2 & +2x_3 & \leq 20 \\
& 2x_1 & +2x_2 & +x_3 & \leq 20 \\
& x_1, & x_2, & x_3 & \geq 0
\end{array}$$

   Suppose you were asked to solve this problem on the midterm exam. You would first transform this into standard form (i.e., you introduce a slack variable for each row), and then use the Simplex Method to find the optimal solution. You would find the following optimal tableau:

		x_1	x_2	x_3	x_4	x_5	x_6
	136	0	0	0	3.6	1.6	1.6
x_3	4	0	0	1	0.4	0.4	-0.6
x_1	4	1	0	0	-0.6	0.4	0.4
x_2	4	0	1	0	0.4	-0.6	0.4

   Unfortunately, you find out that you misread the question: the right hand side of the last constraint should be 40, i.e., the third constraint is $2x_1+2x_2+x_3 \leq 40$ (or with slack variables $2x_1+2x_2+x_3+x_6 = 40$). There is not enough time to repeat your computations from scratch...

   (a) Explain how to read off the inverse basis matrix $\mathbf{B}^{-1}$ from this tableau.

(b) Show how the tableau associated with the current basis changes if you use the new right hand side vector.

(c) Use the dual simplex method to find the optimal solution for the new problem.

4. (Extra Credit) When we talked about the initialization of the Simplex Method, we saw that we can find a feasible solution to a system of linear inequalities (or prove that none exists) by solving a linear program (where the LP is such that finding an initial feasible solution is trivial). So checking feasibility of a system of linear inequalities is not harder than solving a linear programming problem.

   Now, show that the converse is also true: given a linear program in standard form, i.e. $\min c^T x$ s.t. $Ax = b, x \geq 0$, demonstrate a set of linear inequalities such that finding a feasible solution to this set of inequalities is equivalent to solving the linear program.

   You may assume the linear program has a finite optimum. If $A \in R^{m \times n}$ then the system of linear inequalities should have $O(m+n)$ variables and $O(m+n)$ constraints.