Optimization Summer 2011	Homework 5	Khaled Elbassioni Anke van Zuylen

This assignment is **due on June 16** in class. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be written individually without consulting someone else's solutions or any other source like the web.

Problem 1: Consider the uncapacitated network flow problem shown in Figure 1. The label next to each arc is its cost. Solve the problem using the network simplex algorithm. Start with the tree indicated by the dashed arcs in the figure.

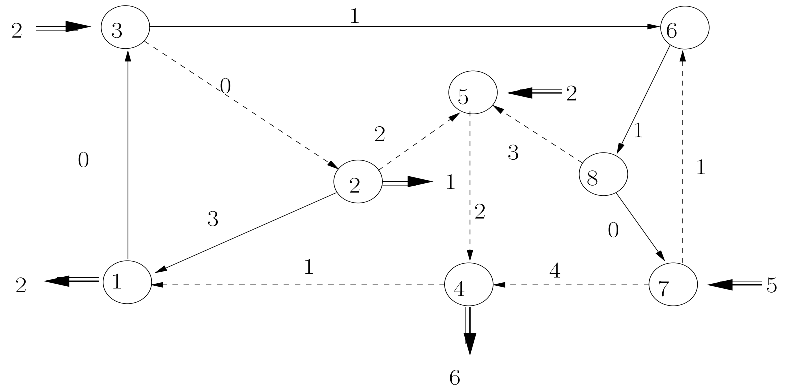

Figure 1: The network flow problem in Exercise 1.

Problem 2: (Equivalence of capacitated network flow and transportation problems) Consider a capacitated network problem defined by a graph $(\mathcal{N}, \mathcal{A})$ and the data u_{ij}, c_{ij}, b_i. Assume that the capacity u_{ij} of every arc is finite. We construct a related transportation problem as follows. For every arc $(i, j) \in \mathcal{A}$, we form a source node in the transportation problem with supply u_{ij}. For every node $i \in \mathcal{N}$, we construct a sink node with demand $\sum_{\{k|(i,k)\in\mathcal{A}\}} u_{ik} - b_i$. At every supply node (i, j) there are two outgoing infinite capacity arcs: one goes to demand node i, and its cost coefficient is 0; the other goes to demand node j and its cost coefficient is c_{ij}. See Figure 2 for an illustration.

Show that there is a one-to-one correspondence between feasible flows in the two problems and that the cost of the two corresponding flows is the same.

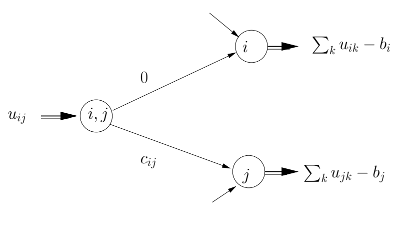

Figure 2: The transportation problem in Exercise 2.

Problem 3: Given an undirected (multi-)graph $G = (V, E)$, with weights $w : V \to \mathbb{Z}$, it is required to orient the edges so as to minimize $\sum_v \max\{w(v) - in(v), 0\}$, where $in(v)$ is the in-degree of v in the orientation. Formulate this problem as a minimum cost network flow problem.

Problem 4: (i) Show that the node-arc incidence matrix of a directed graph is totally unimodular (that is, the determinant of every square submatrix has value in $\{0, \pm\}$).

(ii) Consider the standard LP formulation of the maximum flow problem. Write the dual LP and then use the duality theorem of LP to prove the max-flow min-cut theorem.

Problem 5: (i) **(Vertices of the circulation polytope)** Let $G = (\mathcal{N}, \mathcal{A})$ be a directed graph, with node-arc incidence matrix A. Show that the vertices of the polytope $\{f \in \mathbb{R}^{\mathcal{A}} : Af = 0,\ f \geq 0,\ e'f = 1\}$ (where e is the vector of all ones) are in one-to-one correspondence with the directed cycles of G.

(ii) Let $G = (\mathcal{N}, \mathcal{A})$ be a directed graph with costs $c : \mathcal{A} \to \mathbb{R}$. Consider the problem of finding a directed cycle in G with minimum average cost. Show that this problem can be formulated as a linear program (*Hint*: Use part (i)).

Problem 6: Find the maximum flow from node 1 to node 7 in the network $G = (\mathcal{N}, E)$, where $\mathcal{N} = \{1, \ldots, 7\}$, $E = \{(1,2), (1,3), (2,3), (2,4), (2,5), (3,4), (3,6), (4,5), (4,6), (5,7), (6,5), (6,7)\}$ with arc capacities: $u_{12} = 6$, $u_{13} = 7$, $u_{23} = 1$, $u_{24} = 3$, $u_{25} = 4$, $u_{34} = 2$, $u_{36} = 5$, $u_{45} = 3$, $u_{46} = 2$, $u_{57} = 7$, $u_{65} = 2$, $u_{67} = 4$. Identify the associated minimal cut-set.