Optimization
Summer 2011

Homework 6

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This assignment is **due on June 30** in class. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be written individually without consulting someone else's solutions or any other source like the web.

Problem 1: **(Rotation matrix)** Let μ be an arbitrary vector in $\mathbb{R}^n$ and $e_n = (0, 0, \cdots, 1)^T$. Consider the following matrix:

$$R = 2\frac{(\mu + \|\mu\|e_n)(\mu + \|\mu\|e_n)^T}{\|\mu + \|\mu\|e_n\|^2} - I.$$

Show that

$$R^T R = I \ , \quad R\mu = \|\mu\|e_n.$$

Problem 2: **(Characterization of full-dimensional polyhedra)** A polyhedron $P = \{x \in \Re^n | \ Ax \geq b\}$ is said to be full-dimensional if it has positive volume. Assume that P is bounded and all rows of A are non-zero. Show that the following are equivalent.

(a) The polyhedron P has full dimension.

(b) There exists a point x in P such that $Ax > b$.

(c) There are $n + 1$ extreme points of P that do not lie in a common hyperplane.

Problem 3: Carry out three iterations of the Ellipsoid method towards solving the following LP

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & -3x_1 + 4x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 11 \\ & 2x_1 - x_2 \leq 5 \\ & x_1, x_2 \geq 0. \end{array}$$

Start with the ball $E_0 = E(0, 16I)$. (*Optional:* draw the feasible region and the ellipsoids E_0, E_1, E_2, and E_3).

Problem 4: Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$, where A is an $m \times n$ real matrix and b is an m-dimensional real vector.

- Show that if the rows of A span $\mathbb{R}^n$, then P is nonempty if and only if it has an extreme point.

- Explain how to check if $P = \emptyset$ if the rows of A do not span $\mathbb{R}^n$?

Problem 5: Let $G = (\mathcal{N}, \mathcal{A})$ be a network with n nodes and m arcs, and integral capacities $(u_e : e \in \mathcal{A})$. For two nodes $s, t \in \mathcal{N}$, denote by $\mathcal{P}_{s,t}$ the set of directed paths from s to t in G. Consider the following linear program:

$$\begin{array}{ll} \text{maximize} & \sum_{P \in \mathcal{P}_{s,t}} f_P \\ \text{subject to} & \sum_{e \in P:\ P \in \mathcal{P}_{s,t}} f_P \leq u_e, \quad \forall e \in \mathcal{A}, \\ & f_p \geq 0, \qquad\quad \forall P \in \mathcal{P}_{s,t}. \end{array}$$

- What does this LP compute?
- Show that this LP can be solved in polynomial time, and give an upper bound (as tight as possible) on the running time needed.