

Farkas' Lemma, Dual Simplex and Sensitivity Analysis

1 Farkas' Lemma

Theorem 1. Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Then exactly one of the following two alternatives is true:

- (i) $\exists x \geq 0$ such that $Ax = b$;
- (ii) $\exists y$ such that $y^T A \geq \vec{0}^T, y^T b < 0$.

Proof. Suppose (i) is true. Then (ii) cannot be true, because if both $Ax = b, x \geq 0$ and $y^T A \geq \vec{0}^T$, then $y^T b = y^T Ax \geq 0$.

Now, suppose (i) is not true. To show that (ii) must be true, we consider the following optimization problem

$$\begin{aligned} \max \quad & \vec{0}^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and its dual

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq \vec{0} \end{aligned}$$

If (i) is not true, the primal problem is infeasible. So by strong duality, the dual problem must be either infeasible or unbounded. It cannot be infeasible, because $y = \vec{0}$ is feasible for the dual. So the dual optimum $= -\infty$, and hence there exists some y such that $A^T y \geq \vec{0}, b^T y < 0$. □

2 Dual Simplex Method

Consider an LP in standard form: $\min c^T x$ subject to $Ax = b, x \geq 0$, and its dual $\max y^T b$ subject to $y^T A \leq c^T$. Recall from last week that the (primal) Simplex Method maintains a pair of solutions, one for the primal and one for the dual, that satisfy complementary slackness. If x is a basic solution, let B be the indices of the basic variables. Then the associated dual solution is $y^T = c_B^T B^{-1}$. (Note: this is a basic solution to the dual: a basic solution to $\{y \in \mathbb{R}^m : A^T y \leq c\}$ must have m linearly independent constraints that hold at equality. Indeed, the solution y we define satisfies the constraints $B^T y \leq c_B$ at equality, and the rows of B^T (i.e., the columns of B) are linearly independent.)

Today, we see the dual simplex method, which also maintains a pair of basic solutions, but now, the dual solution is feasible, and we work toward making the primal solution feasible. We use the tableau implementation from the primal simplex method:

$-c_B^T B^{-1} b$	$c^T - c_B^T B^{-1} A$
$B^{-1} b$	$B^{-1} A$

associated solution $x_B = B^{-1} b$
 \Rightarrow

$-c_B^T x_B$	$\bar{c}_1 \dots \bar{c}_n$
$x_{B(1)}$	$B^{-1} A$
\vdots	
$x_{B(m)}$	

- Now, we don't require x to be nonnegative, but we do require $\bar{c} \geq 0$ (which is the same as requiring that the dual solution given by $y^T = \bar{c}_B^T \mathbf{B}^{-1}$ is dual feasible).
- If there is a basic variable $x_{B(\ell)} < 0$, we want to increase it to 0 (make it leave the basis). If all basic variables are ≥ 0 , then we are done, since we have found a feasible primal and dual solution that obey complementary slackness.
- Denote the ℓ -th row of the tableau by $(x_{B(\ell)}, v_1, \dots, v_n)$. Then, we let the following variable enter the basis:

$$j \text{ such that } v_j < 0 \text{ and } \frac{\bar{c}_j}{|v_j|} = \min_{i: v_i < 0} \frac{\bar{c}_i}{|v_i|}.$$

- If such j exists, then we do a pivot as before, i.e., j must become the ℓ -th basic variable, so we do elementary row operations to transform the j -th column of the tableau into $\begin{bmatrix} 0 \\ e_\ell \end{bmatrix}$. Note that this means we add $\frac{\bar{c}_j}{|v_j|}$ times row ℓ to row 0, so the i -th entry in row 0 becomes $\bar{c}_i + v_i \frac{\bar{c}_j}{|v_j|}$ and this is ≥ 0 by the choice of j – we thus maintain dual feasibility.
- Suppose that there is no j such that $v_j < 0$. Then, we can conclude that the primal problem is infeasible (and hence, the dual problem has optimum $+\infty$):

Let g^T be the ℓ -th row of \mathbf{B}^{-1} . Then, the ℓ -th row of the tableau is equal to $g^T [b \quad A]$. We know that $x_{B(\ell)} = g^T b < 0$, and we know that $(v_1, \dots, v_n) = g^T A \geq 0$. Then, by Farkas' Lemma, there exists no $x \geq 0$ such that $Ax = b$.

Example:

0	2	6	10	0	0
2	-2	4	1	1	0
-1	4	-2*	-3	0	1

Since the second basic variable (corresponding to x_5) is less than 0, we want to let x_5 leave the basis. We check which variable should enter the basis by finding $\min_{i: v_i < 0} \frac{\bar{c}_i}{|v_i|} = \min\{\frac{6}{|-2|}, \frac{1}{|0|} - 3\} = 3$, so the corresponding variable (x_2) leaves the basis.

We pivot on element (2, 2) and obtain the following new tableau:

-3	14	0	1	0	3
0	6	0	-5	1	2
1/2	-2	1	3/2	0	-1/2

So when should we use the dual simplex method instead of the primal simplex method?

- It is often easier to find an initial feasible dual solution (for example, if $c, A \geq 0$ then $y = 0$ is a feasible dual solution) – so if we use the dual simplex method, we don't have to do Phase I of the Simplex Method to find an initial bfs.
- In practice, the dual simplex method is faster (see a lecture by Bixby at ADFOCS'03: http://www.mpi-inf.mpg.de/conferences/adfocs-03/Slides/Bixby_2.pdf).
- If we need to resolve an LP, because some of our input data changed, and the previous dual solution remains feasible, e.g.: if we change some b_i or add a new constraint.

3 Sensitivity Analysis

We consider a primal and dual in standard form, i.e.,

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \max & y^T b \\ \text{s.t.} & A^T y \leq c \end{array}$$

Suppose we have solved this problem to optimality, and that the optimal basis is \mathbf{B} , the optimal primal solution is $x_B = \mathbf{B}^{-1}b$, $x_j = 0, j \neq B(1), \dots, B(m)$ and the optimal dual solution is $y^T = c_B^T \mathbf{B}^{-1}b$.

3.1 Changes in b

Suppose we change b to $b + \delta e_i$, i.e., one entry is changed by δ (which may be positive or negative).

The primal solution associated with the current basis has $x_B = \mathbf{B}^{-1}(b + \delta e_i)$. Note that the dual solution remains feasible, and hence the reduced costs remain non-negative. So we only need to check if $\mathbf{B}^{-1}(b + \delta e_i)$ is still feasible, i.e., if it is ≥ 0 .

Suppose it is still feasible, then we have found that the optimal value has changed by

$$c_B^T \mathbf{B}^{-1}(b + \delta e_i) - c_B^T \mathbf{B}^{-1}b = \delta c_B^T \mathbf{B}^{-1} e_i = \delta y^T e_i = \delta y_i.$$

This is the reason why we sometimes call the dual variables the *marginal costs* or *shadow prices* of the requirements: the i -th dual variable tells us by how much the dual changes if we increase the i -th requirement by one unit.

If $\mathbf{B}^{-1}(b + \delta e_i)$ is not feasible, then we can use the Dual Simplex Method from the current basis to find an optimal solution for the new problem.

3.2 Changes in c

Suppose we change c to $c + \delta e_j$, i.e., one entry is changed by δ (which may be positive or negative).

The primal solution associated with the current basis remains feasible, but it may not be optimal, i.e., the reduced cost vector $c^T - c_B^T \mathbf{B}^{-1}A$ may not be non-negative.

We consider two cases:

- Suppose j is a non-basic variable. Then the only reduced cost that changes is $\bar{c}_j \leftarrow \bar{c}_j + \delta$.
- Suppose j is a basic variable, then c_B changes, so all reduced costs are affected.

If the reduced costs are still non-negative, then our current solution is optimal. In the first case (j is non-basic) the objective does not change, in the second case (j is basic) it changes by δ . If some reduced cost becomes positive, we can use the Primal Simplex Method to compute the new optimal solution starting from the current basis.

3.3 Adding a variable

Suppose we add a variable x_{n+1} so our new problem becomes $\min c^T x + c_{n+1}x_{n+1}$ subject to $Ax + A_{n+1}x_{n+1} = b$, $x \geq 0, x_{n+1} \geq 0$.

We can just check if our current basis is still optimal, i.e., if the reduced cost of x_{n+1} is non-negative. We compute

$$\bar{c}_{n+1} = c_{n+1} - c_B^T \mathbf{B}^{-1}A_{n+1}.$$

If it is non-negative, then our current solution is optimal, otherwise, we use the Primal Simplex Method to find an optimal solution starting from the current basis.

3.4 Changes in a column of A

If the column, say A_j is non-basic, then \mathbf{B}^{-1} does not change, so the primal solution is still feasible. The reduced cost of j changes and may become negative; in that case, we can use the primal simplex method to find an optimal solution starting from the current solution.

If the column that changes is basic, the whole tableau changes, in fact, the current basis may not be a basis anymore. You can still say something if the change is small (and the current basis is still a basis), but the arguments become more complicated so we omit the details.

3.5 Adding a constraint

Here, we have to consider two cases: either an inequality or an equality constraint is added.

- Suppose the new constraint is $a_{m+1}^T x \geq b_{m+1}$. If the current optimal solution satisfies this constraint, it is optimal for the new problem. Otherwise, we add a slack variable x_{n+1} and rewrite the constraint as $a_{m+1}^T x - x_{n+1} = b_{m+1}$.

The new constraint matrix is $\begin{bmatrix} A & \vec{0} \\ a_{m+1}^T & -1 \end{bmatrix}$, its right hand side vector is $\begin{bmatrix} b \\ b_{m+1} \end{bmatrix}$ and the new objective coefficients are $\begin{bmatrix} c^T & 0 \end{bmatrix}$.

We can make the new slack variable basic, since the column we added is linearly independent of our current basis. Note that the associated basic feasible solution will have $x_{n+1} < 0$; otherwise, the optimal solution to the original problem was feasible to the new problem. So, we need to use the dual simplex method.

What does the tableau that we start with look like? The new basis matrix is

$$\begin{bmatrix} \mathbf{B} & \vec{0} \\ a^T & -1 \end{bmatrix},$$

where a^T is obtained by keeping only those entries of a_{m+1}^T that correspond to the basic variables. Its inverse is

$$\begin{bmatrix} \mathbf{B}^{-1} & \vec{0} \\ a^T \mathbf{B}^{-1} & -1 \end{bmatrix}.$$

(You can verify that the product of the new basis matrix and its inverse is indeed the $(m+1) \times (m+1)$ identity matrix).

Using this information, we can determine the new tableau, and use the dual simplex method to find the new optimum: The reduced costs become

$$\begin{bmatrix} c^T & 0 \end{bmatrix} - \begin{bmatrix} c_B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{B}^{-1}; & \vec{0} \\ a^T \mathbf{B}^{-1}; & -1 \end{bmatrix} \begin{bmatrix} A; & \vec{0} \\ a_{m+1}^T; & -1 \end{bmatrix} = \begin{bmatrix} c^T - c_B^T \mathbf{B}^{-1} A; & 0 \end{bmatrix}.$$

Row 1, ..., $m+1$, columns 1, ..., $n+1$ of the tableau contain

$$\begin{bmatrix} \mathbf{B}^{-1} & \vec{0} \\ a^T \mathbf{B}^{-1} & -1 \end{bmatrix} \begin{bmatrix} A; & \vec{0} \\ a_{m+1}^T; & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1} A & \vec{0} \\ a^T \mathbf{B}^{-1} A - a_{m+1}^T & 1 \end{bmatrix}.$$

(Note: we thus only need to compute the last row, since the other rows stay the same.)

- Now, suppose we add a new equality constraint $a_{m+1}^T x = b_{m+1}$. If the current optimal solution satisfies this constraint, it is optimal for the new problem.

Otherwise, suppose without loss of generality that our current solution x^* has $a_{m+1}^T x^* > b_{m+1}$. We add an artificial variable $x_{n+1} \geq 0$, and rewrite the constraint as $a_{m+1}^T x - x_{n+1} = b_{m+1}$. Note that we want x_{n+1} to be zero in an optimal solution, since otherwise the solution is not feasible. Hence,

we give x_{n+1} a large objective coefficient M . The new constraint matrix and right hand side are thus the same as in the previous case, $\begin{bmatrix} A & \vec{0} \\ a_{m+1}^T & -1 \end{bmatrix}$, and $\begin{bmatrix} b \\ b_{m+1} \end{bmatrix}$ and the new objective coefficients are $\begin{bmatrix} c^T & M \end{bmatrix}$.

Note that if we set all variables x_1, \dots, x_n according to our current solution, we will have $x_{n+1} = a_{m+1}^T x^* - b_{m+1} > 0$, hence this gives us a primal feasible solution. By the same arguments as in the previous case, this is a basic solution. This solution is not optimal (i.e., the corresponding dual solution is not feasible) if we choose M to be large enough. Hence, we can use the dual simplex method from the current solution.

We now give the details on how to construct the tableau: The inverse basis matrix is given by

$$\begin{bmatrix} \mathbf{B}^{-1} & \vec{0} \\ a^T \mathbf{B}^{-1} & -1 \end{bmatrix},$$

and row $1, \dots, m+1$, columns $1, \dots, n+1$ of the tableau contain

$$\begin{bmatrix} \mathbf{B}^{-1} & \vec{0} \\ a^T \mathbf{B}^{-1} & -1 \end{bmatrix} \begin{bmatrix} A; & \vec{0} \\ a_{m+1}^T; & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1} A & \vec{0} \\ a^T \mathbf{B}^{-1} A - a_{m+1}^T & 1 \end{bmatrix}.$$

(Note: we thus only need to compute the last row, since the other rows stay the same.)

The new reduced costs are given by

$$\begin{bmatrix} c^T & M \end{bmatrix} - \begin{bmatrix} c_B^T & M \end{bmatrix} \begin{bmatrix} \mathbf{B}^{-1}; & \vec{0} \\ a^T \mathbf{B}^{-1}; & -1 \end{bmatrix} \begin{bmatrix} A; & \vec{0} \\ a_{m+1}^T; & -1 \end{bmatrix} = \begin{bmatrix} c^T - c_B^T \mathbf{B}^{-1} A - M(a^T \mathbf{B}^{-1} A - a_{m+1}^T); & M \end{bmatrix}.$$

Example:

$$\begin{array}{llllll} \min & -5x_1 & -x_2 & +12x_3 & & \\ \text{s.t.} & 3x_1 & +2x_2 & +x_3 & & = 10 \\ & 5x_1 & +3x_2 & & +x_4 & = 16 \\ & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

The optimal tableau is

12	0	0	2	7
2	1	0	-3	2
2	0	1	5	-3

Note that $\mathbf{B}^{-1} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.

We add the constraint $x_1 + x_2 = 5$, which is violated by the optimal solution $(2, 2, 0, 0)$.

We form the new problem

$$\begin{array}{llllll} \min & -5x_1 & -x_2 & +12x_3 & & -Mx_5 \\ \text{s.t.} & 3x_1 & +2x_2 & +x_3 & & = 10 \\ & 5x_1 & +3x_2 & & +x_4 & = 16 \\ & x_1 & +x_2 & & & -x_5 = 5 \\ & x_1, & x_2, & x_3, & x_4, & x_5 \geq 0 \end{array}$$

We want to obtain the tableau associated with the solution $(2, 2, 0, 0, 1)$.

The new inverse basis matrix is

$$\begin{bmatrix} \mathbf{B}^{-1} & \vec{0} \\ a^T \mathbf{B}^{-1} & -1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 5 & -3 & 0 \\ 2 & -1 & -1 \end{bmatrix},$$

and we compute $a^T \mathbf{B}^{-1} A - a_{m+1}^T = [0 \ 0 \ 2 \ -1]$. to obtain entries $1, \dots, n$ of the new row added to the tableau, and the new reduced costs are obtained from the previous ones by subtracting $M(a^T \mathbf{B}^{-1} A - a_{m+1}^T) = M [0 \ 0 \ 2 \ -1]$.

So our new tableau is

12	0	0	2-2M	7+M	0
2	1	0	-3	2	0
2	0	1	5	-3	0
1	0	0	2	-1	1