Optimization I — Summer 2011 — Lecture 8

Linear Programming Duality

1 Taking a dual

Consider the following LP:

$$\begin{array}{llllll} \min & 6x_1 & +4x_2 & +2x_3 & \\ \text{s.t.} & 4x_1 & +2x_2 & +x_3 & \geq 5 \\ & x_1 & +x_2 & & \geq 3 \\ & & x_2 & +x_3 & \geq 4 \\ & & & x_i & \geq 0, \text{ for } i = 1,2,3. \end{array}$$

Suppose you want to give a lower bound on the optimal value of this LP. What can we say?

- Since all variables are non-negative, $6x_1 + 4x_2 + 2x_3 \geq 4x_1 + 2x_2 + x_3$. So the value of the LP must be at least 5.
- By the same argument, $6x_1 + 4x_2 + 2x_3 \geq (4x_1 + 2x_2 + x_3) + 2(x_1 + x_2) \geq 5 + 2 \cdot 3 = 11$.
- Again by the same argument, $6x_1+4x_2+2x_3 \geq (4x_1+2x_2+x_3)+(x_1+x_2)+(x_2+x_3) \geq 5+3+4 = 12$.

How do we determine the best lower bound we can achieve this way? By setting up a different LP! Let y_1 be the number of times we take the first constraint, y_2 the number of times we take the second constraint and y_3 the number of times we take the third constraint. Then the lower bound we get is $5y_1 + 3y_2 + 4y_3$, and we need to ensure that this is a lower bound, i.e.

$$6x_1 + 4x_2 + 2x_3 \geq y_1(4x_1 + 2x_2 + x_3) + y_2(x_1 + x_2) + y_3(x_2 + x_3).$$

We can do this by ensuring that $4y_1 + y_2 \leq 6$ (since we have $6x_1$ in the objective value, and $4x_1$ in the first constraint, $1x_1$ in the second constraint and $0x_1$ in the third constraint), $2y_1 + y_2 + y_3 \leq 4, y_1 + y_2 \leq 2$. Also, we need to have $y_1, y_2, y_3 \geq 0$ (otherwise the inequalities in the constraints change direction, and we would not get a lower bound). We thus get the following LP for getting the best lower bound:

$$\begin{array}{llllll} \max & 5y_1 & +3y_2 & +4y_3 & \\ \text{s.t.} & 4y_1 & +2y_2 & & \leq 6 \\ & 2y_1 & +y_2 & +y_3 & \leq 4 \\ & y_1 & & +y_3 & \leq 2 \\ & & & y_i & \geq 0, \text{ for } i = 1,2,3. \end{array}$$

This is called the *dual* linear program.

Let's consider a second example:

$$\begin{array}{lllll} \min & x_1 & +2x_2 & +3x_3 & \\ \text{s.t.} & -x_1 & +3x_2 & & = 5 \\ & 2x_1 & -x_2 & +3x_3 & \geq 6 \\ & & & x_3 & \leq 4 \\ & x_1 \geq 0, & x_2 \leq 0, & x_3 \text{ free.} & \end{array}$$

Let's again find the best possible lower bound on the objective value: We let y_1, y_2, y_3 again be the number of times we take the first, second and third constraint respectively. The lower bound on the optimal value we want to obtain is $5y_1 + 6y_2 + 4y_3$.

We need to ensure that

$$x_1+2x_2+3x_3 \geq y_1(-x_1+3x_2)+y_2(2x_1-x_2+3x_3)+y_3(x_3) = (-y_1+2y_2)x_1+(3y_1-y_2)x_2+(3y_2+y_3)x_3,$$

and that

$$y_1(-x_1+3x_2)+y_2(2x_1-x_2+3x_3)+y_3(x_3) \geq 5y_1+6y_2+4y_3.$$

First, we derive constraints by using the sign constraints of x_j and comparing the coefficient of each x_j on the left hand side and the right hand side of the first inequality:

$$\begin{array}{rcl}
\text{know: } x_1 \geq 0 \;\; \& \;\; \text{want: } x_1 \geq (-y_1+2y_2)x_1 & \Rightarrow & -y_1+2y_2 \leq 1 \\
\text{know: } x_2 \leq 0 \;\; \& \;\; \text{want: } 2x_2 \geq (3y_1-y_2)x_2 & \Rightarrow & 3y_1-y_2 \geq 2 \\
\text{know: } x_3 \text{ free} \;\; \& \;\; \text{want: } 3x_3 \geq (3y_2+y_3)x_3 & \Rightarrow & 3y_2+y_3 = 3
\end{array}$$

Then, we derive sign constraints for the variables y_i by using the i-th constraint of the linear program and comparing the coefficient of each y_i on the left hand side and the right hand side of the second inequality:

$$\begin{array}{rcl}
\text{know: } -x_1+3x_2 = 5 \;\; \& \;\; \text{want: } y_1(-x_1+3x_2) \geq 5 & \Rightarrow & y_1 \text{ free} \\
\text{know: } 2x_1-x_2+3x_3 \geq 6 \;\; \& \;\; \text{want: } y_2(2x_1-x_2+3x_3) \geq 6y_2 & \Rightarrow & y_2 \geq 0 \\
\text{know: } x_3 \leq 4 \;\; \& \;\; \text{want: } y_3(x_3) \geq 4y_3 & \Rightarrow & y_3 \leq 0
\end{array}$$

So, we get the following dual linear program

$$\begin{array}{llllll}
\max & 5y_1 & +6y_2 & +4y_3 & \\
\text{s.t.} & -y_1 & +2y_2 & & \leq 1 \\
 & 3y_1 & -y_2 & & \geq 2 \\
 & & 3y_2 & +y_3 & = 3 \\
 & y_1 \text{ free}, & y_2 \geq 0 & y_3 \leq 0. &
\end{array}$$

We can now give the general rules for finding the dual of a given linear program:

Primal	minimize	maximize	Dual
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

For a problem in standard form, we thus find the following pair of primal and dual problem:

$$\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & Ax = b \\
 & x \geq 0
\end{array}
\qquad\qquad
\begin{array}{ll}
\max & y^T b \\
\text{s.t.} & A^T y \leq c
\end{array}$$

There are a few questions we could ask:

- **What happens if we take a dual linear program, multiply its objective by -1 to obtain a minimization problem, and then take the dual of this LP?**

  Suppose we use the dual from our previous example, where we replace the objective $\max 5y_1+6y_2+4y_3$ by $\min -5y_1-6y_2-4y_3$. We also replace y_i by $\bar{x}_i$ and we will denote the variables in the dual by $\bar{y}_j$.

$$\begin{array}{llllll}
\min & -5\bar{x}_1 & -6\bar{x}_2 & -4\bar{x}_3 & \\
\text{s.t.} & -\bar{x}_1 & +2\bar{x}_2 & & \leq 1 \\
 & 3\bar{x}_1 & -\bar{x}_2 & & \geq 2 \\
 & & 3\bar{x}_2 & +\bar{x}_3 & = 3 \\
 & \bar{x}_1 \text{ free}, & \bar{x}_2 \geq 0, & \bar{x}_3 \leq 0. &
\end{array}$$

The dual is:

$$\begin{array}{llllll} \max & \bar{y}_1 & +2\bar{y}_2 & +3\bar{y}_3 & \\ \text{s.t.} & -\bar{y}_1 & +3\bar{y}_2 & & = -5 \\ & 2\bar{y}_1 & -\bar{y}_2 & +3\bar{y}_3 & \leq -6 \\ & & & \bar{y}_3 & \geq -4 \\ & \bar{y}_1 \leq 0, & \bar{y}_2 \geq 0, & \bar{y}_3 \text{ free .} & \end{array}$$

Now, note that if we let $\bar{y}_j = -x_j$, then we get

$$\begin{array}{llllll} \max & -x_1 & -2x_2 & -3x_3 & \\ \text{s.t.} & x_1 & -3x_2 & & = -5 \\ & -2x_1 & +x_2 & -3x_3 & \leq -6 \\ & & & -x_3 & \geq -4 \\ & x_1 \geq 0, & x_2 \leq 0, & x_3 \text{ free.} & \end{array}$$

We can multiply each of the constraints by -1 and replace the objective by $\min x_1 + 2x_2 + 3x_3$ to see that the dual of the dual is the primal LP!

- **Do we get the same dual linear program if we take the dual directly, and if we first convert a problem into standard form, and then take the dual?**

  For example, we can take the dual of the following LP directly:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \text{ free} \end{array} \qquad\qquad \begin{array}{ll} \max & y^T b \\ \text{s.t.} & A^T y = c \\ & y \geq 0 \end{array}$$

  or, we can change it into standard form, by replacing x by $x^+ - x^-$ and by adding surplus variables, and then take its dual:

$$\begin{array}{ll} \min & c^T x^+ - c^T x^- \\ \text{s.t.} & Ax^+ - Ax^- - Is \geq b \\ & x^+ \geq 0, x^- \geq 0, s \geq 0 \end{array} \qquad\qquad \begin{array}{ll} \max & y^T b \\ \text{s.t.} & A^T y \leq c \\ & -A^T y \leq -c \\ & -Iy \leq 0 \\ & y \geq 0 \end{array}$$

  It is easy to see that the dual linear programs we derived in these two ways are the same.

We state the following two theorems which give the general answers to these two questions without proof (as the proof is just a tedious exercise).

Definition 1. *Two linear programs are equivalent if they are either both infeasible, or they are both feasible and have the same objective value.*

Theorem 1. *If we transform the dual linear program into an equivalent minimization problem, and take its dual, then we obtain a problem that is equivalent to the original problem.*

Theorem 2. *If we transform a minimization linear program into another equivalent minimization problem , then their duals are also equivalent.*

We will use the convention that the original LP, which we will call the *primal LP* is a minimization problem, and that the dual LP is a maximization problem.

2 Duality Theorem

We derived the dual by trying to find the best possible lower bound on the objective value of the primal LP. In fact, we constructed the dual so that the objective value of a feasible solution to the dual LP gives a lower bound on the objective value of *any* feasible solution to the primal LP, as we prove for LPs in standard form:

Theorem 3 (Weak duality)**.** *If x is a feasible solution to the primal LP and y is a feasible solution to the dual LP, then $c^T x \geq b^T y$.*

Proof. We may restrict our attention to a primal LP in standard form, by Theorem 2. Then, we have

$$c^T x \geq (A^T y)^T x = y^T (Ax) \geq y^T b, \tag{1}$$

where the first inequality follows because $A^T y \geq c$ and $x \geq 0$, and the second inequality because $Ax = b$. $\square$

Note that it is thus the case that:

- If the optimal value of the primal LP is $-\infty$, then the dual LP is infeasible.
- If the optimal value of the dual LP is $+\infty$, then the primal LP is infeasible.
- If x is feasible to the primal LP and y is feasible to the dual LP, and $c^T x = b^T y$, then x is optimal for the primal and y is optimal for the dual LP.

In fact, we can prove the following result, which is the main result on linear programming duality.

Theorem 4 (Strong duality)**.** *If a linear program has an optimal solution, then so does its dual and the respective objective values are equal.*

Proof. By Theorem 2, it suffices to consider a problem in standard form, i.e., $\min\{c^T x : Ax = b, x \geq 0\}$. The dual is given by $\max\{b^T y : A^T y \leq c\}$.

Suppose the rows of A are linearly independent, and there exists an optimal solution (if the rows are not linearly independent, we can first transform it into an equivalent problem by removing redundant constraints). The simplex algorithm algorithm (with an appropriate pivoting rule so that cycling is avoided) will find an optimal basic feasible solution, say x, and Let A_B be the corresponding basis matrix. Note that $c^T x = c_B^T x_B = c_B^T A_B^{-1} b$.

The reduced costs are all non-negative, so $\bar{c}^T = c^T - c_B^T A_B^{-1} A \geq 0$. We let $y^T = c_B^T A_B^{-1}$. Then $c^T \leq y^T A$, or $A^T y \leq c$. So y is a feasible solution to the dual LP. Also, $b^T y = y^T b = c_B^T A_B^{-1} b = c^T x$. $\square$

We find the following possibilities for the outcome of a linear program and its dual:

Primal \ Dual	Finite optimum	Unbounded	Infeasible
Finite Optimum	possible	impossible	impossible
Unbounded	impossible	impossible	possible
Infeasible	impossible	possible	possible