

THE PROBABILISTIC METHOD AND RANDOMIZED ALGORITHMS

Assignment 6 — Due on July 26, 2011

1. Let $A_1, \dots, A_n \subseteq \{1, \dots, m\}$ with $\sum_{i=1}^n 2^{1-|A_i|} < 1$. Prove there exists a two-colouring $\chi : \{1, \dots, m\} \rightarrow \{0, 1\}$ with no A_i monochromatic. With $m = n$ give a deterministic algorithm to find such a χ in polynomial time.
2. Show that the following program **Much-ado-about-nothing** terminates with probability 1. What is the expected number of recursive calls of the subroutines **Beeper**, **Lover** and **Spammer** altogether (i.e. the number of beeps + love messages + spam e-mails)? What is the expected number of “I love you” messages displayed on the screen?

```
Proc much-ado-about-nothing
  Call Beeper;
```

```
Proc Beeper
  Beep for 10 seconds;
  N := Poisson trial of parameter 4;
  For i from 1 to N
    Call Lover;
```

```
Proc Lover
  Print "I love you";
  For i from 1 to 1000
    Call Spammer with probability 1/1000;
```

```
Proc Spammer
  Send a spam message to enemies;
  k := Outcome of a fair die roll;
  do while k=6
    Call Beeper;
    k := Outcome of a fair die toss;
  end do;
```

3. Prove that for any fixed $\epsilon > 0$ and $0 < p < 1$ there is an $m_0 = m_0(\epsilon, p)$ so that for every $n > 2m > m_0$, the probability that two fixed disjoint sets A and B , each of size m , of the random graph $\mathcal{G}(n, p)$ do not form an ϵ -regular pair is at most ϵ .
4. * (Removal lemma) Prove that for any fixed graph H on h vertices and for any $\epsilon > 0$, there is a $\delta = \delta(\epsilon, H) > 0$ such that if an n -vertex graph G is ϵ -far from being H -free (in the dense graph model), then G contains at least δn^h copies of H .

5. Let H be a fixed graph on h vertices. Use Problem 2 to prove that the graph property of H -freeness (under dense graph model) is testable with query complexity depending only on ϵ .