Topological Methods in Discrete Geometry

Summary of Lecture 1 MPI, Summer 2011

Radon's theorem states that any set P of $(d+2)$ points in $\mathbb{R}^d$ can be partitioned into two sets, say P_1 and P_2, such that $\texttt{conv}\ (P_1) \cap \texttt{conv}\ (P_2) \neq \emptyset$.

This statement can be re-stated as follows: given any affine map $f : \partial\Delta^{d+1} \to \mathbb{R}^d$, there exist two points $x_1, x_2 \in \partial\Delta^{d+1}$ with disjoint supports (i.e., the simplices that contain x_1 and x_2 are disjoint) and where $f(x_1) = f(x_2)$.

The above two statements are equivalent: the image, under f, of each simplex is the convex-hull of the vertices of the simplex. So two intersecting convex-hulls on disjoint points corresponds to two disjoint simplices whose image intersects.

Topological Radon's theorem removes the "affine" constraint for the mapping: given any map $f : \partial\Delta^{d+1} \to \mathbb{R}^d$, there exist two points $x_1, x_2 \in \partial\Delta^{d+1}$ with disjoint supports (i.e., the simplices that contain x_1 and x_2 are disjoint) and where $f(x_1) = f(x_2)$. So a strict generalization of Radon's theorem.

Proof technique: Lets say, for contradiction, that there is a 'bad' map, i.e., a continuous map f with the property that for every two points x_1, x_2 with disjoint supports, $f(x_1) \neq f(x_2)$. Then if such a map f exists, simply extend it to get the product map $f_{pair} : X \to Y$, where

$$\begin{aligned} X &= (\partial\Delta^{d+1} \times \partial\Delta^{d+1}) \setminus \{(x_1, x_2) \mid supp(x_1) \cap supp(x_2) = \emptyset\} \\ Y &= (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(y_1, y_2) \mid y_1 = y_2\} \end{aligned}$$

and $f_{pair}(x_1, x_2) \to (f(x_1), f(x_2))$.

Now simply have to show that f_{pair} does not exist. It is easier to prove the more general problem that no map from the space X to the space Y exists. Unfortunately, that is not true for a general such map; so we have not sufficiently modelled the problem, as f_{pair} is a map of a very specific type from X to Y. In particular, f_{pair} applies the *same function* f to both the first and second coordinate given to it. How to capture that property at a general level?

A weak way to capture it, which fortunately is sufficient in this particular case, is by endowing both X and Y with antipodality structure in such a way that a general function from X to Y will not be 'antipodality preserving', while f_{pair} will be. And then try to prove the (weaker) statement that there is no antipodality-preserving map from X to Y (w.r.t. the antipodality structure we have given X and Y).

We can equip a space X with antipodality structure by defining a function $\nu : X \to X$, where ν defines a homeomorphism from X to X, and $\nu^2(x) = x$. Then the space (X, ν) is called a $\mathbb{Z}_2$-space.

In particular, for both X and Y, define the antipodal of the point $(a, b) \in X$ (or Y) to be the point $(b, a) \in X$ (or Y). Observe that this is a very generic natural way to give product-spaces an antipodality structure. One can check that this satisfies all the required properties of antipodality.

Then f_{pair} satisfies antipodality (i.e., antipodal pairs in X map to antipodal pairs in Y). This is because $f_{pair}((x_1, x_2)) \to (f(x_1), f(x_2))$, and so $f_{pair}((x_2, x_1)) \to (f(x_2), f(x_1))$. An antipodality-preserving map from X to Y, where both X and Y are equipped with antipodality structure, is called a $\mathbb{Z}_2$-map.

Now it only remains to show that there is no antipodality-preserving map (i.e., a $\mathbb{Z}_2$-map) from X to Y. That is done by computing an invariant, the $\mathbb{Z}_2$-index, over both X and Y. I will leave the definition of $\mathbb{Z}_2$-index, and its properties to be read from the book, where it is explained nicely.

In particular, it follows from the definition of $\mathbb{Z}_2$-index that for a $\mathbb{Z}_2$-map to exist from X to Y, $Ind_{\mathbb{Z}_2}(X) \leq Ind_{\mathbb{Z}_2}(Y)$. But straightforward computation shows that for the case $d = 1$, $Ind_{\mathbb{Z}_2}(X) = 1$, while $Ind_{\mathbb{Z}_2}(Y) = 0$. And we've proven topological Radon's theorem for the case $d = 1$.

X and Y are called 'Deleted Products' of $\partial\Delta^{d+1}$ and $\mathbb{R}^d$ respectively. It is easy to see that $Ind_{\mathbb{Z}_2}(Y) = d - 1$; it is hard (in a technical sense) to compute the $\mathbb{Z}_2$-index of X, which is the reason we only did it for the simple case of $d = 1$.