Topological Methods in Discrete Geometry

Summary of Lecture 3 MPI, Summer 2011

So far, we have defined the antipodality structure ($\mathbb{Z}_2$-action) on topological spaces embedding in some $\mathbb{R}^d$. But note that all the underlying structures were simplicial complexes. So now we define an analog of the $\mathbb{Z}_2$-action on a simpicial complex K, such that the affine extension of this will give a $\mathbb{Z}_2$-action on the corresponding polyhedron $||K||$.

Analog of $\mathbb{Z}_2$-action for Simplicial Complexes

The analog of $\mathbb{Z}_2$-action for a simplicial complex K is simply a function $f : V(K) \longrightarrow V(K)$ such that

- f is a matching (though a vertex can be matched to itself), and
- The mapping f is simplicial. In other words, if $\{v_1, \ldots, v_k\}$ is a simplex in K, then $\{f(v_1), \ldots, f(v_k)\}$ is also a simplex in K.

We can now extend f is the standard way to a $\mathbb{Z}_2$-action on $||K||$: a point $x \in ||K||$ can be written as a (unique) convex combination of it's support, and so $f(x)$ maps x to the same convex combination of the simplex defined by the image vertices of its support. In particular,

$$x = \sum_{i=1}^{k} \alpha_i v_i \Longleftrightarrow f(x) = \sum_{i=1}^{k} \alpha_i f(v_i)$$

Note: in fact, apart from giving a pairing of vertices of K, f also gives a pairing of simplices of K: if v_i pairs with $f(v_i)$, then the simplex defined by $\{v_1, \ldots, v_k\}$ pairs with the simplex defined by $\{f(v_1), \ldots, f(v_k)\}$.

Example 1: last time, we looked at the cross-polytope in $\mathbb{R}^d$, which is the convex-hull of the standard basis $\{e_1, -e_1, \ldots, e_d, -e_k\}$. The $\mathbb{Z}_2$-action on the cross-polytope was defined by '$-$' ($x \leftrightarrow -x$). It can be seen that if we look at the cross-polytope as a simplicial complex, and define $f(e_i) = -e_i$, then the affine extension of this f gives the same $\mathbb{Z}_2$-action as before.

Example 2: We defined the $\mathbb{Z}_2$-action on the embedding of the join $||K * K||$ by $t \cdot \psi_1(x) + (1-t) \cdot \psi_2(y) \Longleftrightarrow (1-t) \cdot \psi_1(y) + t \cdot \psi_2(x)$. Now, define $f(\cdot)$ on $K * K$ by pairing up the two copies of the same vertex (in the first and second copy of K). Again, the affine extension of f gives the same $\mathbb{Z}_2$-action as before.

Example 3: We now know the pairing for the complex $K * K$. But what about the complex $K * L$? There we need the function $f : V(K) \longrightarrow V(K)$ for K, and the function $g : V(L) \longrightarrow$

$V(L)$ for L. Then the function $h : V(K * L) \longrightarrow V(K * L)$ is simply $f \sqcup g$. In other words, as the join $K * L$ is on the vertex set $V(K) \dot\cup V(L)$, we can simply keep the same respective antipodality functions on the vertices as before. The first condition (matching) is clear; I leave the easy verification of the second condition (simplicial map) to the reader.

Finally, we have to look at $\mathbb{Z}_2$-maps – recall $\mathbb{Z}_2$-maps from $||K||$ to $||L||$ preserve the $\mathbb{Z}_2$-action – and make sure appropriate generalizations work for simplicial complexes. In particular, if there is a simplicial map $h : V(K) \to V(L)$ which preserves antipodality of vertices ($f : V(K) \to V(K)$, $g : V(L) \to V(L)$), then an affine extension of it should correspond to be a $\mathbb{Z}_2$-map from $||K||$ (with $\mathbb{Z}_2$-action $||f||$) to $||L||$ (with $\mathbb{Z}_2$-action $||g||$). Indeed that is the case, as can be verified.

Order Complexes

Given a simplicial complex K, we can derive another complex $\Delta(K)$ (called the *order complex* of K) from it as follows: each simplex of K (which can be thought of as a subset of vertices $V(K)$) becomes a vertex, and there is a simplex between vertices $v_1', \ldots, v_k'$ in $\Delta(K)$ iff the sets corresponding to v_i' form a chain under the partial order induced by '$\subseteq$'.

When K is a simplicial complex, $\Delta(K)$ is also a simplicial complex; in fact, we also have $||K|| \simeq ||\Delta(K)||$. Even if K is just a set of simplices over the base vertex set $V(K)$, $\Delta(K)$ is *still* a simplicial complex.

As noted before, the affine extension of the function $f : V(K) \to V(K)$ gives a pairing of simplices. It can also be shown that this map is simplicial. So this gives a map f' satisfying the two conditions for $\Delta(K)$.

Sarkaria's Inequality

It is straightforward, and can be read from the book. Will fill in more details here later.

Van Kampen-Flores Theorem

It is straightforward, and can be read from the book. Will fill in more details here later.