Problem Set 3

Topological Methods in Geometry

SS 2011

Problem 1 (Borsuk Ulam in small dimensions).
The Borsuk Ulam theorem states that for any continous map $f : \mathbb{S}^d \mapsto \mathbb{R}^d$, $\exists x \in \mathbb{S}^d$ s.t. $f(x) = f(-x)$. We have already seen how to prove this for $d = 1$ by using the intermediate value theorem. Prove the statement for $d = 2$.
Hint: Let $\pi : \mathbb{R}^2 \mapsto \mathbb{R}$ be a function such that for any $p = (x, y) \in \mathbb{R}^2, \pi(p) = x$. Consider two antipodal points, say a and $-a$ on $\mathbb{S}^2$ such that $\pi(f(a)) \neq \pi(f(-a))$. Prove that on any curve connecting a and $-a$ on $\mathbb{S}^2$ there is a point x s.t. $\pi(f(x)) = \pi(f(-x))$. What can you say about the set of all such points?

Problem 2 (Homeomorphisms).
Prove that $\mathbb{R}$ is homeomorphic to $(0, 1)$. Prove that $\mathbb{S}^d \setminus \{x\}$, where $x \in \mathbb{S}^d$, is homeomorphic to $\mathbb{R}^d$. Assume the subspace topology inherited from the Euclidean topology in all cases.

Problem 3 (Lyusternik-Shnirelman theorem).
Prove that the following statement is equivalent to the statement of the Borsuk Ulam theorem.
(LS-c) If $F_0, \ldots, F_n$ are closed sets covering $\mathbb{S}^n$, then one of these sets contains a pair of antipodal points of $\mathbb{S}^n$.

Problem 4 (Intersecting colorful triangles).
Prove that given any set of two red, two blue and two green points in the plane, there always exist two colorful triangles with disjoint vertices which intersect each other. A colorful triangle is a triangle with one red, one blue and one green vertex.
Hint: Consider a map from the boundary of the three dimensional cross polytope to the plane so that the six vertices are mapped to the six specified points in the plane. The cross polytope is $\{x \in \mathbb{R}^3 : \|x\|_1 \leq 1\}$ and its boundary is homeomorphic to $\mathbb{S}^2$.