# Problem Set 4

## Topological Methods in Geometry

SS 2011

**Problem 1** (KKM Lemma in two dimensions)**.**
Let $\sigma$ be a triangle with vertices $v_0, v_1$ and $v_2$. Let $C_0$, $C_1$ and $C_2$ be closed sets such that for each triple $i, j, k \in \{0, 1, 2\}$, not necessarily distinct, the union of the sets $C_i, C_j$ and $C_k$ contains the simplex spanned by $v_i, v_j$ and $v_k$. Prove that $C_0, C_1$ and $C_2$ have a common intersection.

**Problem 2** (Tucker Lemma in two dimensions)**.**
Let $T$ be a triangulation of $\mathbb{S}^1$ which is centrally symmetric ($\sigma \in T \implies -\sigma \in T$ for any simplex $\sigma$). Let $\lambda : V(T) \mapsto \{-1, +1, -2, +2\}$ be a labelling function such that $\lambda(-v) = -\lambda(v)$ for each vertex $v$ in $T$. Show that if there is no edge $(u, v) \in T$ such that $\lambda(u) = -\lambda(v)$, then the number of edges in $T$ whose vertices have labels $-1$ and $+2$ is odd. Use this to prove Tucker's lemma in two dimensions.

**Problem 3** (Jordan Curve Theorem for Polygons)**.**
Let $P$ be a simple closed polygon in the plane. Let $K = \mathbb{R}^2 \setminus P$. Prove that there are two points $a$ and $b$ in $K$ which cannot be joined by a polygonal path in $K$. Prove that any other point in $K$ can be joined to either $a$ or $b$ by a polygonal path in $K$.

**Problem 4** (Hex Theorem)**.**
Let $R$ be an axis parallel box (a rectangle with the interior included) in the plane. Let $T$ be a (geometric) triangulation of $R$ i.e. $\|T\| = R$. A path in the triangulation is a sequence of vertices $v_1, \ldots, v_{k+1}$ such for any $i \in \{1, \ldots, k\}$, $v_i$ and $v_{i+1}$ span a 1-simplex (an edge) of $T$. Assume that the vertices of $T$ are colored red and blue. Prove that either there is a red path (a curve on which all vertices are red) joining the left and right sides of the rectangle or there is a blue path joining the top and bottom sides of the rectangle.
*Hint:* Use Brouwer's Fixed Point theorem.