Problem Set 6

Topological Methods in Geometry

SS 2011

Problem 1 (Lusternik Schnirelmann Theorem for Closed and Open Sets)**.**
Prove that if $F_1, \dots F_{d+1}$ are sets covering $\mathbb{S}^d$, where each F_i is either open or closed then one of the sets contains a pair of antipodal points of $\mathbb{S}^d$.
Hint: Try to replace the closed sets by open sets containing it. If a closed set does not contain a pair of antipodal points, its diameter is strictly smaller than 2.

Problem 2 (Ham-Sandwich Theorem for Point Sets)**.**
In the lecture, we proved the Ham-Sandwich theorem for measures s.t. any hyperplane has measure zero by any of the measures. Prove Ham-Sandwich theorem for point sets in $\mathbb{R}^d$, i.e. given d finite sets of points in $\mathbb{R}^d$, show that there exists a hyperplane such that both the closed halfspaces defined by the hyperplane contain at least half the points (rounded up if there are odd number of points) of each set. Assume that the set of all of the given points is in general position.

Problem 3.
Show that given three finite sets of points in the plane, there exists either a circle or a line so that each of the closed sets defined by it contain at least half of the points of each set.
Hint: Replace each point $p = (x, y)$ in the plane by the point $(x, y, x^2 + y^2)$ in $\mathbb{R}^3$.

Problem 4.
Let $I = [-1, 1]$. Let α be a continuous curve joining the left and right sides of I^2 and let β be a continuous curve joining the top and bottom sides of I^2. Both curves lie within I^2. Prove that α and β intersect.
Hint: Think of the curves as functions $\alpha, \beta : I \mapsto I^2$. For $\gamma \in \{\alpha, \beta\}$, define functions $\gamma_x, \gamma_y : I \mapsto I$ s.t. $\gamma(t) = (\gamma_x(t), \gamma_y(t))$. Define a function $F : I^2 \mapsto I^2$ as follows:

$$F(s,t) = \frac{(\beta_x(t) - \alpha_x(s), \alpha_y(s) - \beta_y(t))}{\|(\beta_x(t) - \alpha_x(s), \alpha_y(s) - \beta_y(t))\|_\infty}$$