Universität des Saarlandes
FR 6.2 Informatik

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Exercises for Graph Theory

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss11/graph_theory/

Assignment 13 Deadline: Thursday, July 14, 2011

Rules: The first problem serves as a preparation for the test conducted in the exercise class. You should solve it, but you do not need to hand it in. The test yields 8 points. The remaining problems have to be handed in *nicely written up as you would do in a thesis, as the Diestel does, ...* in the Thursday lecture. These homework problems yield 8 points in total.

You need to collect at least 50% of all these points (tests and written homework) from (i) the first three exercise sessions, (ii) the first six sessions, and (iii) the whole term.

Occasionally, there might be bonus problems, which yield additional bonus points. They are, typically, more difficult or not that closely related to that week's content of the lecture.

This is the last exercise sheet of the course, and therefore the last opportunity to collect points.

Exercise 1 (*oral homework, in total 8 points via test*)

(Re-)read and understand the material in Sections 11.1, 11.2, and 11.4 in the Diestel book.

Exercise 2 (*written homework, 2 points*)

Let the random variable X denote the number of copies of K_4 in a random graph $G_{n,p}$.

Compute the expectation and the variance of X, and use the methods of first and second moment to derive a threshold result similarly to the one we derived for K_3 in the lecture.

Exercise 3 (*written homework, 3 points*)

Let $G = (V, E)$ be a connected graph with $|V| = n \geq 3$ and $|E| = m$. In this exercise you will use the probabilistic method to show that G contains an independent set of size at least $\frac{n^2}{4m}$.

Set $d := 2m/n$. Note that d is exactly the average degree in G, and that $d > 1$ because G is connected with more than two vertices. Let $S \subseteq V$ be a random subset of vertices obtained by including each vertex with probability $1/d$ independently. Let the random variable X denote the size of S, and let the random variable Y denote the number of edges with both endpoints in S.

a) Show that for any outcome of the probabilistic experiment described, G contains an independent set of size at least $X - Y$. [1P.]

b) Calculate the expectation of $Z := X - Y$ as a function of n and d. [1P.]

c) Infer that G contains an independent set of size at least $\frac{n}{2d} = \frac{n^2}{4m}$. [1P.]

Exercise 4 (*written homework, 3 points*)

Let the random variable Y denote the number of isolated vertices in a random graph $G_{n,p}$.

a) Compute the *expectation* of Y, and apply the first moment method to infer that for any $\varepsilon > 0$ and any $p = p(n) \geq (1+\varepsilon)\frac{\ln(n)}{n}$, the random graph $G_{n,p}$ does not contain any isolated vertices a.a.s. [1.5P.]

b) Compute the *variance* of Y, and apply the second moment method to infer that for any $\varepsilon > 0$ and any $p = p(n) \leq (1-\varepsilon)\frac{\ln(n)}{n}$, the random graph $G_{n,p}$ contains an isolated vertex a.a.s. [1.5P.]

(Hint: Recall that $1 - x \leq \mathrm{e}^{-x}$ for all $x \in \mathbb{R}$. This is a good estimate for small values of x; in particular, we have $1 - x \geq \mathrm{e}^{-x-x^2}$ for $0 \leq x < 1/2$.)

Remark: Note that this is a stronger threshold behaviour than what we saw in the lecture for the triangle example (where we proved asymptotic probabilities of 0 or 1 under the assumption that $p = p(n)$ was larger or smaller than $1/n$ *in order of magnitude*). One says that $p_0(n) = \frac{\ln(n)}{n}$ is a *sharp threshold* for the property of not containing an isolated vertex.

Bonus exercise on next page!

Exercise Bonus 1 (*4 points*)

Consider the following number guessing game. One player, called Carole for simplicity, thinks of a number between 1 and n. The other player, called Paul, tries to find this number by asking arbitrary yes/no questions, which Carole truthfully answers. It is clear that Paul, if he is clever, needs at most $\lceil \log_2(n) \rceil$ questions until he knows the secret number.

Note that, crucially for the following, the minimum number of questions Paul needs to determine Carole's number with certainty does not change if we allow Carole to play a 'devil's strategy', that is, if we allow her to change her number to some other number which is consistent with her previous answers at any time. We will therefore give Carole this freedom throughout the following.

a) Use the probabilistic method to prove that the $\lceil \log_2(n) \rceil$ bound is tight. Assume that Paul asks exactly q questions according to an arbitrary fixed strategy, and that Carole answers these questions *randomly*, i.e., regardless of Paul's questions she answers 'yes' or 'no' with probability $1/2$ each time, independently of previous questions and answers.

   (i) Prove that if the number of questions satisfies $q < \log_2(n)$, with positive probability there is more than one number that is not ruled out by Carole's answers. (Hint: Compute the expected number of remaining possible numbers after q questions.)

   (ii) Infer from (i) that Paul has no deterministic strategy to determine Carole's number with fewer than $\log_2(n)$ questions. [2P.]

b) Now, let us slightly change the rules of the game and assume that Carole is allowed to lie exactly once. Consequently, a number become impossible only after two answers say that it is not among the valid numbers. Clearly, Paul can determine the number with $2\lceil \log_2(n) \rceil$ questions using his previous strategy, but asking every question twice. However, one can show that he can do much better and then needs only about $\log_2(n) + \log_2(\log_2(n))$ questions.

   Adapt the probabilistic argument from (a) to show that he cannot do better than this: Show that Paul has no deterministic strategy to determine Carole's number with fewer than $\log_2(n) + \log_2(\log_2(n))$ questions. [2P.]