Universität des Saarlandes

FR 6.2 Informatik

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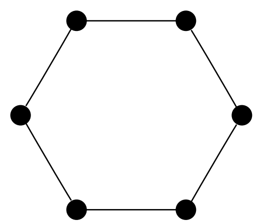
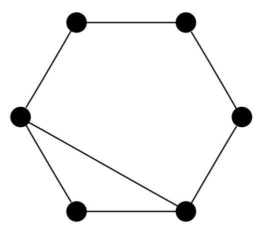
Solution for Exercise 4

Exercise 1

b) The following graph on six vertices has a minimum cardinality vertex cover of size 3.



c) The following graph on six vertices is non-bipartite (due to the odd length cycle) and has a minimum cardinality vertex cover of size 3.



d) It's $n-1$. If we select at most $n-2$ vertices, there is an uncovered edge between the two vertices which are not in the cover.

e) It's $\lfloor n/2 \rfloor$. There is an edge between any pair of vertices, so we can match a maximal number of pairs. If the number of vertices is odd, one vertex must remain unmatched.

f) The red matching is maximal with respect to subset inclusion, but it doesn't have maximum cardinality.

Exercise 3 We prove that a given matching $M \subseteq E$ is *not* maximum-cardinality if and only if there is an augmenting path w.r.t. M in G.

The 'if' direction is easy: if there is an augmenting path P w.r.t. M in G, then $M' := M \triangle E(P)$ is a matching in G with $|M'| > |M|$.

Conversely, let M', $|M'| > |M|$, be a larger matching in G. We show that then G contains an augmenting path w.r.t. M.

Let G' be the graph spanned by the edges of $M \triangle M'$. Since every vertex of G' has at most two edges incident to it (one from M and one from M'), each connected component of G' is either a cycle or a path. Moreover, all cycles in G' are of even length because otherwise we would have two edges of either M or M' adjacent to each other, violating the matching property. Thus M and M' have the same number of edges in cycles of G'.

Now, since $|M'| > |M|$ there must be a connected component in G' which is a path, say P, and contains more edges from M' than from M. Since P alternatingly uses edges from M' and M, it must start and end with edges from M'. Furthermore, as P is a connected component of G', its endvertices must be unmatched in M. Hence P is an augmenting path w.r.t. M.

Exercise 4

Let G be a bipartite graph with partition classes A and B such that

$$\forall S \subseteq V : |N(S)| \geq |S| - d$$

for some fixed $d \in \mathbb{N}$. We construct a graph $G' = (A\dot{\cup}(B\dot{\cup}D), E(G) \cup \hat{E})$, where D is a set of d new vertices (not in G), and $\hat{E}$ is the set of all possible edges between vertices of A and vertices of D (i.e., we have a complete bipartite graph between A and D).

In G', we have $|N(S)| \geq |S| - d + d = |S|$ for all subsets $S \subseteq V$. Therefore, by Hall's theorem, there is a matching M of A in G'. Since $|D| = d$, there are at most d vertices in A which are matched to vertices in D. Consequently there are at least $|A| - d$ vertices which are matched to vertices in B. By removing all edges from M that are incident to a vertex of D, we thus get a matching of size $|A| - d$ in G, as required.

Exercise 5

If suffices to show that any $r \times n$ Latin Square with $r < n$ can be extended to an $(r+1) \times n$ Latin Square. So, assume that we have an $r \times n$ Latin Square S, and construct the following bipartite graph: $G = (A \cup B, E)$ where $A = \{(s_{11}, s_{21}, ..., s_{r1}), (s_{12}, s_{22}, ..., s_{r2}), ..., (s_{1n}, s_{2n}, ..., s_{rn})\}$, $B = \{1, 2, ..., n\}$, $E = \{\{(s_{1i}, s_{2i}, ..., s_{ri}), j\} : j \in [n] \setminus \{s_{1i}, s_{2i}, ..., s_{ri}\}\}$. Note that G is a bipartite graph with partition classes A and B.

The vertices of A correspond to columns of the given Latin Square and the vertices in B correspond to numbers from 1 to n. If there is an edge from $a \in A$ to $b \in B$ then the column that corresponds to a can be extended with b. Note that matchings of G correspond exactly to possible ways of extending the Latin square in row $r+1$.

Every vertex in A has $n-r$ neighbors, since there are already r different numbers in the given column of S. Moreover also every vertex of B has $n-r$ neighbors, since every number occurs exactly once in every row and at most once in every column.

Hence, G is a $(n-r)$-regular bipartite graph, and therefore has a 1-factor. This 1-factor yields a possible filling of the $(r+1)$th row.

Exercise 6

Player 2 has a winning strategy if and only if G contains a perfect matching.

If G has a perfect matching, we describe a winning strategy for player 2, otherwise, we give a winning strategy for player 1.

Suppose G has a perfect matching. Player 2 fixes one of these, say $M = \{e_1, \ldots, e_{n/2}\}$, $e_i = \{u_i, v_i\}$. Now, whenever player 1 picks w.l.o.g. u_i, then player 2 immediately picks v_i. Clearly, player 2 can always pick such a vertex and so wins the game.

Now, suppose G does not have a perfect matching. Then player 1 fixes a maximum-cardinality matching, say M, and picks a vertex v which is *not* matched by M in his first step. After that, whenever player 2 picks a vertex $u \in e$ matched by M, player 1 immediately picks the other endpoint of e.

So, the only way player 2 can win the game is by choosing a vertex u that is not matched by M. But then the path formed by all vertices picked so far is an augmenting path from v to u, contradicting the maximality of M. Hence this cannot happen, and what we described is indeed a winning strategy for player 1.