Universität des Saarlandes
FR 6.2 Informatik

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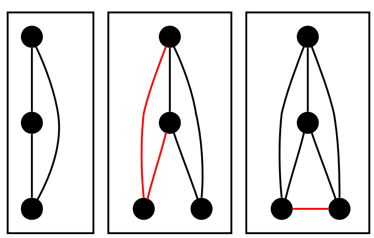
Solution for Exercise 5

Exercise 1 (*oral homework, total 8 points via test*)

a) The easy direction in Tutte's Theorem is that for any $S \subseteq V(G)$ we have $|S| \geq q(G-S)$, where $q(G-S)$ is the number of odd sized components in $G-S$. This direction follows because given a subset $S \subseteq V(G)$, at least one vertex in each odd sized component of $G-S$ must be matched to a vertex in S.

b) —

c) Here is one way to do this:



d) This can be shown by assuming for the sake of contradiction that G is not 2-connected. A case distinction on the possibilities for a cut vertex in G leads to a contradiction to either the 2-connectedness of H, or to the fact that P is an H-path.

e) Suppose $B(G)$ contains a cycle. As $B(G)$ is bipartite, this cycle must have even length (Proposition 1.6.1), and so it must contain at least two blocks. Let $C = b_1 a_1 b_2 a_2 \ldots a_k b_1$ denote this cycle, where a_i is a cut vertex, and b_i a block, for all $i \in \{1, \ldots, k\}$. Then we can easily construct a cycle in G as $a_1 P_2 a_2 P_3 \ldots a_k P_1 a_1$, where P_i is a path in the block b_i. This contradicts the lemma we saw in the lecture (Lemma 3.1.2 in the Diestel) which states that every cycle in G must be contained in exactly one block.

f) Each edge of a tree is a bridge, hence it is also a block. Thus the blocks of a tree are precisely its edges.

Exercise 2 (*2 points*) Let $S \subseteq V(G)$, and take C as an odd sized component of $G-S$. As G is $(2k+1)$-regular, the sum of the vertex degrees in C is odd, but only an even number is contributed to the sum by edges contained in C. That is, we have

$$\sum_{v \in C} \deg_G(v) = |C| \cdot (2k+1) = 2|E(C)| + |\{\{c,s\} : c \in C, s \in S\}|.$$

From this it follows that $|\{\{c,s\}|c \in C, s \in S\}|$ is odd, and because G is $2k$ edge connected, it must be that $|\{\{c,s\} : c \in C, s \in S\}| \geq 2k+1$. Therefore, the number of edges between S and $G-S$ must be at least $(2k+1) \cdot q(G-S)$. However, it is also at most $(2k+1)|S|$ due to the regularity of G. Thus

$$(2k+1) \cdot q(G-S) \leq (2k+1)|S|$$

and so $q(G-S) \leq (2k+1)|S|$. Tutte's condition therefore applies, and G has a perfect matching.

Exercise 3 (*2 points*)

a) Let $e = \{u,v\}$ be an edge of G, let G' be the graph that results from subdividing e, and let x be the new vertex. Suppose there is a cut vertex z in G'. Clearly $z \neq x$, as removing x from G' is equivalent to removing the edge e in G. As G is 2-connected, we know that $\lambda(G) \geq \kappa(G) = 2$ (Proposition 1.4.2 in the Diestel), and hence $G-e$ is connected.

   Therefore there must be a vertex y in $G'-z$ that is unreachable from x. But then y must also be unreachable from at least one of u or v (both, if $z \notin \{u,v\}$) and therefore z is a cut vertex for G as well, a contradiction to the assumption that G is 2-connected.

b) Subdividing any edge in a tree increases the number of edges in G and, by exercise 1f), also the number of blocks in G.

Exercise 4 (*4 points*)

a) If every pair u, v of vertices lies on a cycle C, there are at least two independent paths P, Q between u and v along the cycle such that $C = P \cup Q$. Removing a vertex $x \notin \{u,v\}$ can only affect at most one of those paths, as $P \cap Q = \{u,v\}$, therefore no cut vertex exists.

b) Suppose G is two connected, and let $e = \{u,v\}$ be an edge of G such that u and v do not lie on a cycle. As we know that $\lambda(G) \geq \kappa(G)$ (Proposition 1.4.2 in the Diestel), we can easily derive a contradiction. Removing e must disconnect u and v, as otherwise there would be a path that avoids e, yielding a cycle in G that contains u and v. But then $\lambda(G) = 1$ and G cannot be 2-connected.

c) Let $u, v \in V(G)$. The proof is by induction on the distance $d_G(u,v)$ of u and v in G. If $d_G(u,v) \leq 1$, the claim follows from subproblem (b). For the inductive step, let $d_G(u,v) \geq 2$, and let $P = u, w, \ldots, v$ be a shortest $u-v$ path in G. By the induction hypothesis, u and w belong to some cycle of G, and so there are two independent $w-v$

paths Q_1 and Q_2 in G. As G is 2-connected, there must be a $u-v$ path R that avoids w. Let r be a vertex of R that belongs also to either Q_1 or Q_2, and is closest to u in R (see figure below). If no such vertex exists, uwQ_1v and R are two independent paths between u and v, and thus uwQ_1vRu is a cycle. Otherwise, let r lie w.l.o.g. on Q_1. Then the two paths $uRrQ_1v$ and uwQ_2v are independent, and form a cycle.