Universität des Saarlandes
FR 6.2 Informatik

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Solution for Exercise 6

Exercise 1 *(oral homework, in total 8 points via test)*

(a) Read carefully and understand Menger's Theorem (local (3.3.1) and global (3.3.6) versions) in Diestel's book.

(b) Let G be a graph, and let $A, B \subseteq V(G)$. If there is an A-B separator in G of size k, can there be more than k disjoint A-B paths in G? Why?

Solution: No, because if there would be more than k disjoint A-B paths in G, then removing any set of k vertices in G would leave at least one of these paths intact, and would therefore not separate A from B.

(c) If X is an A-B separator in G, is any A-X separator in G also an A-B separator of G? Why?

Solution: Yes, since if X is an A-B separator in G, then each A-B path uses a vertex from X. An A-X separator separates all A-X paths, hence all A-B paths, since each A-B path uses at least one vertex from X.

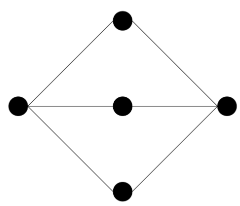
(d) What is the graph obtained by contracting an arbitrary edge in K_5?

Solution: K_4

(e) Read carefully and understand all the notions and definitions given in Section 4.2 of Diestel's Book.

(f) Draw a planar drawing of $K_{2,3}$.

Solution:



(g) What are the faces of a plane graph?

Solution: A face of a graph G is a maximal connected subset of $\mathbb{R}^2 \setminus G$.

(h) What is Euler's Formula?

Solution: Let G be a connected plane graph with n vertices, m edges, and ℓ faces. Then $n - m + \ell = 2$.

Exercise 2 *(written homework, 2 points)*

Suppose that G_1 and G_2 are k-connected graphs with $V(G_1) \cap V(G_2) \geq k$. Prove that $G = G_1 \cup G_2$ is also k-connected.

Solution:

The statement obviously holds for $k = 0$. Assume therefore that $k \geq 1$. We show that for any set $S \in V(G)$ with $|S| < k$, any pair of distinct vertices in $G - S$ are connected by a path. So let S be an arbitrary such subset, and pick two distinct vertices u and w in $G - S$. If both u and w are in G_1, or both are in G_2, then we know that they are connected in $G - S$ by the assumption that G_1 and G_2 are k-connected. Otherwise, since $|V(G_1) \cap V(G_2)| \geq k$, there will remain at least one vertex of $V(G_1) \cap V(G_2)$ in the graph $G - S$, pick one and call it v. Since $G_1 - S$ and $G_2 - S$ are connected, there is a path P_1 from u to v in G_1, and a path P_2 from v to w in G_2. Hence, there is a walk uP_1vP_2w from u to w in $G - S$, which means that u to w are connected in $G - S$.

Exercise 3 *(written homework, 3 points)*

Let G be a k-connected graph for some $k > 1$.

(a) Show that for any cycle C in G, and any vertex $v \in V(G) \setminus V(C)$, there are $\min\{k, |V(C)|\}$ paths from v to $V(C)$ such that none of these paths intersect except on v. [1P]

Solution: Let $\ell = min\{k, |V(C)|\}$, and let $A = N(v)$, $B = V(C)$, and X be a set of vertices separating A and B in G. Then $|A| \geq k \geq \ell$, since G is k-connected, and $|B| \geq \ell$ by definition. Furthermore, G is ℓ-connected, since G is k-connected and $\ell \leq k$. Thus, by the ℓ-connectedness of G, it follows that $|X| \geq \ell$. Applying Menger's Theorem (Theorem 3.3.1) with the sets A and B above, we get that there are at least ℓ disjoint A-B paths in G. Connecting v to each of these paths at their endpoints in A gives us the required paths.

(b) Use (a) to show that any set of k vertices $X \subseteq V(G)$ is contained on some cycle of G. (Hint: Start with a cycle that contains as many vertices of X as possible, and use (a) to arrive at a contradiction if this cycle does not contain all of X). [2P]

Solution:

Consider a cycle $C := v_1, \ldots v_c v_1$ that contains as many vertices of X as possible (we know one exists since G is k-connected and $k > 1$). Assume that there is a vertex $w \in X \backslash V(C)$. By (a), there are $min\{k, |V(C)|\}$ paths from w to $V(C)$ that do not intersect except in v. If $|V(C)| \leq k$, we know that there are paths from w to v_1 and from w to v_c that only intersect in w. But then $v_1 C v_c w v_1$ is a cycle with more vertices from X than C, and we get a contradiction. Otherwise, we consider the case where $|V(C)| > k$. We know that there can be at most $k - 1$ vertices from X in $V(C)$, and that there are k paths from w to $V(C)$ that do not intersect except in w. Therefore, by the pigeonhole principle, two of those paths P_1 and P_2 end in two different vertices in $V(C)$, say in v_i and in v_j, such that there is no vertex from X between them on C. Hence, the cycle $wP_1v_iCv_jP_2w$ has more vertices from X than C, and again we get a contradiction.

Exercise 4 *(written homework, 3 points)*

Show that any graph can be drawn in $\mathbb{R}^3$ with no edge crossings (in the same sense that any planar graph can be drawn in the plane with no edge crossings).

Solution:

Let $G = (\{v_1 \dots v_n\}, \{e_1 \dots e_m\})$ be an arbitrary graph. Put each vertex $v_i \in V(G)$ on the point $(0, 0, i)$, and draw each edge $e_j \in E(G)$ on a different half plane $\alpha_j := \{(x, y, z) : x \in \mathbb{R}^+, z \in \mathbb{R}, \text{and } y = j \cdot x\}$ such that it intersects with the z-axis only at its endpoints. Since each edge is drawn on a different plane and meets the z-axis only at its endpoints, and moreover the half planes intersect only at the z-axis, any pair of distinct edges can only intersect at their endpoints. Thus the above construction gives a drawing in $\mathbb{R}^3$ with no edge crossings.