Universität
des
Saarlandes
FR 6.2 Informatik

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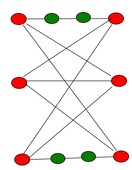
Solution for Exercise 7

Exercise 1 oral homework, total 8 points via test

(a) —

(b) The following graph is a subdivision of $K_{3,3}$ that contains 10 vertices. Green vertices are the subdividing vertices.



(c) A graph X is a minor of a graph G if G contains an inflation of X as a subgraph.

(d) True. We proved in the lecture that every minor X of G is also a topological minor of G whenever $\Delta(X) \leq 3$ (Proposition 1.7.4 in Diestel). As $\Delta(K_{3,3}) = 3$, the claim follows.

(e) (Corollary 1.7.3 in Diestel) Let X and G be finite graphs. Then X is a minor of G iff there are graphs $G_0,\ldots,G_n$ such that $G_0 = G$ and $G_n = X$ and each G_{i+1} arises from G_i by deleting an edge, contracting an edge, or deleting a vertex.

(f) In the proof of Lemma 4.4.3, we contract an edge xy in our given graph G, and let v_{xy} be the new vertex. For the proof to follow we need all neighbors of v_{xy} to lie on some cycle in G/xy. To get this, we use the fact that since G/xy is 3-connected, $G/xy - v_{xy}$ is 2-connected, and so by proposition 4.2.6 all neighbors of v_{xy} in G/xy lie on a cycle.

We need the assumption that G does not contain K_5 or $K_{3,3}$ as minors, and not as topological minors, because we want to contract an arbitrary edge xy in G. We use the fact that if G/xy has a minor X, then G will also have X as its minor. Since we are contracting an arbitrary edge, this holds only for minors and not necessarily for topological minors.

Exercise 2

(a) Consider a set of graphs $\mathcal{G}$ which is closed under minors, and let $\mathcal{F}$ denote the set of minor-minimal graphs which are not in $\mathcal{G}$. Then $\mathcal{F}$ is antichain in the minor order, and so it is finite by Robertson and Seymour's Graph Minor Theorem. To complete the proof, we show that $\mathcal{F}$ is a forbidden minor characterization of $\mathcal{G}$.

Let G be an arbitrary graph. If G contains a minor F for some $F \in \mathcal{F}$, then $G \notin \mathcal{G}$ since otherwise we would have $F \in \mathcal{G}$ by the fact that $\mathcal{G}$ is closed under minors; a contradiction to the definition of $\mathcal{F}$. Conversely, if G does not contain any minor $F \in \mathcal{F}$, then it must be that $G \in \mathcal{G}$, since $\mathcal{F}$ is the set of all minor-minimal graphs which do not belong to $\mathcal{G}$. It therefore follows that $\mathcal{F}$ is a forbidden minor characterization of $\mathcal{G}$.

(b) Consider a set of graphs $\mathcal{G}$ which is closed under minors, and let $\mathcal{F}$ be a finite forbidden minor characterization of $\mathcal{G}$ (which exists according to (a)). We need to show that $\mathcal{G}$ also has a finite forbidden *topological* minor characterization. For this, as $\mathcal{F}$ is finite, it is enough to show that for any $F \in \mathcal{F}$ there exists a finite set of graphs $\mathcal{H}(F)$ such that F is minor of G iff H is a topological minor of G for some $H \in \mathcal{H}(F)$.

So fix some arbitrary $F \in \mathcal{F}$. For each $x \in V(F)$, let $\mathcal{T}_x$ denote the set of all trees with at most $d_H(x)$ leafs, and with no internal vertices of degree 2. Then each tree in $\mathcal{T}_x$ has at less than $2 \cdot d_H(x)$ vertices, and so $\mathcal{T}_x$ is finite. Now consider a graph H obtained by replacing each vertex $x \in V(F)$ with a tree $T_x \in \mathcal{T}_x$, and connecting two leafs of T_x and T_y whenever $xy \in E(F)$. By construction we have

$$H \text{ is a topological minor of } G \Rightarrow F \text{ is a minor of } G.$$

Furthermore, any inflation of F contains as a topological minor a graph H that can be constructed as above. Thus, letting $\mathcal{H}(F)$ denote the set of all such graphs H, we have

$$F \text{ is a minor of } G \iff H \text{ is a topological minor of } G \text{ for some } H \in \mathcal{H}(F)$$

as required. Since $\mathcal{T}_x$ is finite for each $x \in V(F)$, we get that $\mathcal{H}(F)$ is finite, and we are done.

Exercise 3

(a) Let G be an outerplanar graph, and let $\widetilde{G}$ denote a planar drawing of G with all vertices lying on the boundary of the outer face. Consider the graph H obtained from G by adding a new vertex v which is adjacent to all vertices of G. Since all vertices of G lie on the boundary of the outer face f in $\widetilde{G}$, we can connect any point p in f to all vertices of $\widetilde{G}$ by Jordan curves which intersect each other only at p. In this way, we can extend $\widetilde{G}$ to a planar drawing of H, and so H is planar.

Conversely, let G be a graph such that the graph H obtained by adding a vertex v that is connected to every vertex in G is planar. Consider some planar drawing $\widetilde{H}$ of H. Since v is connected to all the vertices of G, all these vertices lie on some face in $\widetilde{H}$. Now let $\widetilde{G}$ denote the drawing obtained from $\widetilde{H}$ by removing v along with all of its incident edges. Then $\widetilde{G}$ is a planar drawing of G. By using an appropriate mapping of $\widetilde{G}$ to the sphere and then back again to the plane (*e.g.* using stereographic projections), we get another planar drawing of G in which all vertices lie on the outer face. Hence, G is outerplanar.

(b) Let G be an outerplanar graph. If we add a vertex v that is connected to every vertex in G, then $G + v$ is a planar graph (by (a)). We need to show that G does not contain K_4 nor $K_{2,3}$ as topological minors. We will prove this by contradiction. Suppose G contains K_4 as a topological minor. Then G has a subgraph X which is a subdivision of K_4, and $X + v$ contains a subdivision of K_5. Hence, by Kuratowski's theorem, $G + v$ is not planar; a contradiction. A similar contradiction can be obtained also for $K_{2,3}$. Thus, G does not contain K_4 nor $K_{2,3}$ as topological minors.

Conversely, let G be a graph that does not contain K_4 nor $K_{2,3}$ as topological minors. Then the graph H obtained by adding a vertex to G which is adjacent to all $V(G)$ does not contain K_5 as a topological minor, nor $K_{3,3}$ as a topological minor. Thus, H is planar by Kuratowski's theorem, and so G is outerplanar by (a).