Universität des Saarlandes
FR 6.2 Informatik

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Solution for Exercise 11

Exercise 2 (*2 points*)

Let G be a perfect graph, and let H be an arbitrary induced subgraph of G. Since H is an induced subgraph of a perfect graph, we have $\omega(H) = \chi(H)$. Consider an arbitrary coloring of H with $\chi(H)$ colors. The maximum number of vertices that can be colored with the same color is $\alpha(H)$. Hence, the maximum number of vertices that can be colored with $\chi(H)$ colors is $\alpha(H) \cdot \chi(H)$. Thus $|V(H)| \leq \alpha(H) \cdot \chi(H) = \alpha(H) \cdot \omega(H)$.

Exercise 3 (*4 points*)

(a) Let G be a chordal graph. If G is complete then the statement is trivial, so assume G is not complete, and in particular, that $|V(G)| > 1$. Let $a, b \in V(G)$ be two non-adjacent vertices in G, and let $X \subseteq V(G) \setminus \{a, b\}$ be a minimal a-b separator. Also, let $s, t \in X$. By minimality of X, there are s-t-paths both in G_1 and in G_2. Let P_1 be a shortest s-t-path in G_1, and let P_2 be a shortest s-t-path in G_2. Then sP_1tP_2s is a cycle of length at least 4 in G, and as G is chordal, there must be some edge in G connecting two non-incident vertices on this cycle. This edge cannot be between an internal vertex of P_1 and an internal vertex of P_2, because X is a separator, and it cannot be between two internal vertices of P_1, nor between two internal vertices of P_2, due to the minimality of P_1 and P_2. Thus, s and t must be adjacent, and so X induces a complete subgraph in G.

(b) Let G be a chordal graph. We prove the statement by induction in $|V(G)|$. If $|V(G)| = 1$, the statement is trivial, so assume $|V(G)| > 1$, and that the statement is true for all graphs with fewer vertices than G. Suppose G is not complete. Let $a, b \in V(G)$ be two non-adjacent vertices in G, and let $X \subseteq V(G) \setminus \{a, b\}$ be a minimal a-b separator. Then X induces a complete subgraph in G according to (a). Let A denote the connected component of $G - X$ containing a, and let B be the connected component of $G - X$ containing b. By induction, $A \cup G[X]$ is either complete or it has two non-adjacent simplicial vertices. In both cases, this implies that there is a vertex s_A in A which is simplicial $A \cup G[X]$, since there are no two nonadjacent vertices in X. Furthermore, as s_A has no neighbors in G outside of $A \cup G[X]$, s_A is simplicial also in G. Thus, by a similar argument we can obtain a vertex $s_B \in V(B)$ which together with s_A forms a pair of non-adjacent simplicial vertices of G.

(c) Let G be a chordal graph. The proof is by induction on $V(G)$. From (b), it follows that G has a simplicial vertex x. By induction, since $G - v$ is chordal and has fewer vertices than G, it has a simplicial ordering $(v_1, \ldots, v_n)$. Then $(v_1, \ldots, v_n, x)$ is a simplicial ordering for G.

(d) To prove the claim we show that any graph G admitting a simplicial ordering has $\chi(G) \leq \omega(G)$. This suffices, since every chordal graph admits a simplicial ordering according to (c), and since a graph admits a simplicial ordering iff all of its induced subgraphs admit a simplicial ordering. Assume then that G admits a simplicial ordering $(v_1, \ldots, v_n)$, and let N_i denote the set of neighbors of v_i in $G[\{v_1, \ldots, v_i\}]$, for each $i \in \{1, \ldots, n\}$. As any maximal complete subgraph of G is induced by $v_i \cup N_i$ for some $i \in \{1, \ldots, n\}$, we get that $|N_i| < w(G)$ for all $i \in \{1, \ldots, n\}$. Thus $col(G) \leq w(G)$, and as $\chi(G) \leq col(G)$ (Proposition 5.2.2 in the Diestel), we have $\chi(G) \leq w(G)$.

Exercise 4 (*2 points*)

Let S be a maximal set of disjoint A-B-paths, and let $X := \bigcup_{P \in S} V(P)$. By maximality of S, the vertex set X is an A-B-separator in G, and so X is infinite by the assumption in the exercise. But then, as $V(P)$ is finite for every $P \in S$, it must be the case that S is infinite.