2 con / 2 edge con.

- planarity, graph drawing, embeddings of planning
- Traffic control, reliable network design

(Open) ear decompositions

Lemma: Let H be a proper subgraph with > 1 vertex of a block G. Then G contains an open ear of H.

Proof: - if H is spanning ($V(H) = V(G)$), open ear $e \in G \setminus H$

- if H is not spanning

deletion of vw $\Rightarrow$ open ear of H

Def. Let B be a set of base graphs, let O be a finite set of graph operations, then

Def. A - construction sequence of G is sequence $\{G_0, G_1, \dots, G_k\}$, where $G_k = G$, under condition that G_{i+1} is obtained by operation from O and $G_0 \in B$.

Def. An open) ear decomposition is a construction sequence with B - set of cycles and every operation adding an ear / open ear.

Thm. 1) Graph G is 2 con. $\Leftrightarrow$ G has an open ear decomposition.

2) Graph G is 2 edge con. $\Leftrightarrow$ G has an ear decomposition.

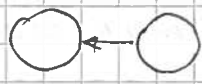
1. "$\Leftarrow$"  coffee-mug lemma;

"$\Rightarrow$"  by previous lemma;

Application: Strong orientation / Robbin 1939
[if directed graph is strongly connected]

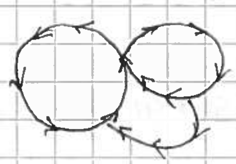
Thm: A connected graph G has strong orientation ⟺ G has no bridges.

" ⟹ "



Linear time!

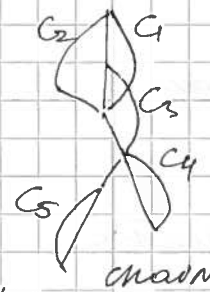
" ⟸ "



ear decomposition → direct edges and obtain strong orientation.

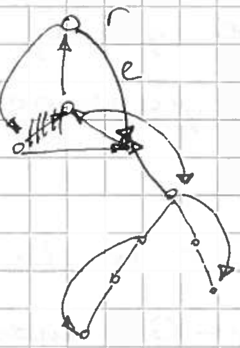
Testing 2 (edge) con. & Finding (open) ear decomposition

- Do DFS on G, get DFS-tree with root r and DFI



chain decomposition

⟸



- Decompose G into chains (path / cycle)
- For every vertex $v \in V$ assign DFI order
- For every back edge e starting at v
  - traverse C(e) [— unique cycle with only one backedge e] until first vertex is already visited
  - call traversed subgraph a chain!

$\mathcal{C} = \{C_1, C_2, \dots, C_{m-n+1}\}$

Thm 1: G is 2 edge-connected ⟺ $\mathcal{C}$ partitions E.

Thm 2: G is 2 connected ⟺ $\mathcal{C}$ partitions E s.t. C_1 is the only cycle in $\mathcal{C}$.

Lemma 1 edge e is bridge ⟺ e is not contained in $\mathcal{C}$.

Def. T(x)

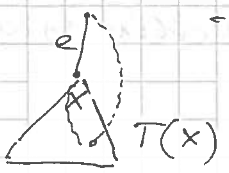
▷ Assume $n > 2$.

" ⟹ " assume that e is contained in $\mathcal{C}_i$ ⟹
let b is be the unique back edge in $\mathcal{C}_i$

C(b

↯ bridges are not in cycles

" ⟸ " - e is not in a chain. x = end point of e farthest from r → e is tree-edge



- There is no back edge with exactly one end point in T(x
→ e is bridge.

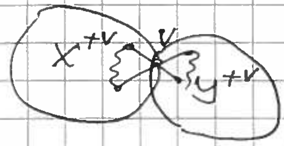
Lemma 2: Assume that min degree $\delta(G) \geq 2$.

Then v is a cut vertex $\Leftrightarrow$ v is incident to a bridge
or
v is first vertex of a cycle in $\mathcal{C} \setminus C_1$

▷ "$\Rightarrow$" assume v is cut-vertex

and v is not incident to a bridge.

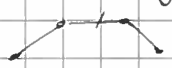
Let x and y be of $G \setminus \{v\}$ $\Rightarrow$



- X^{+v} and Y^{+v} - subgraphs of G induced by $X \cup \{v\}$ and $Y \cup \{v\}$.
- X, Y contain ≥ 2 neighbors of v

$\Rightarrow$ X^{+v}, Y^{+v} contain cycles containing v

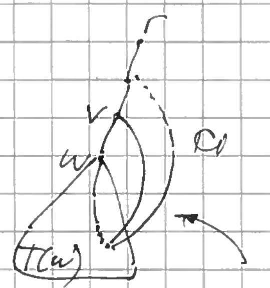
- C_1 exists, w.log. say X^{+v} does not contain C_1

"$\Leftarrow$" - if  v

- let v be first vertex of cycle in $\mathcal{C} \setminus C_1$

- if $v = r$ $\Rightarrow$ r is cut-vertex
- if $v \neq r$ $\Rightarrow$ then no backedge starts at a vertex with smaller DFI as v and ends in $T(w)$

$\Rightarrow$ v is a cut-vertex.



How to compute all cut-vertices and all bridges

$O(n+m)$ time by looking at chains

Block-Cut-Trees

$O(n+m)$ (Exercise)

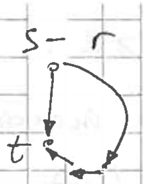
How to compute (open) ear decomposition (C_1 - cycle)

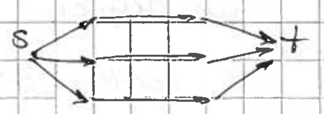
$O(n+m)$ if G is 2-con. $\Rightarrow$ every chain C_i will be an open ear

if G is 2-edge-con $\Rightarrow$ every chain C_i will be an ear.

Def. A bipolar orientation is an acyclic orientation with exactly one source ($\text{indeg}(s) = 0$) and exactly one sink ($\text{outdeg}(t) = 0$).

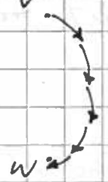
[Exist only for 2-con graphs]





For every C_i : orient C_i accordingly;

$(v-w)$ ear



<u>Def</u>: st-numbering: Top sort from bipolar orientation.