(Geometric) dual

**Prop.**

G be a planar emb.

The dual graph G* of G consists of
- a vertex f* for each face
- an edge e* = fᵢ* fᵢ* ∈ E(G*) for all fᵢ*, fᵢ* sharing one edge.

![Diagram of a dual graph](image)

- 1 bijection edge ↔ dual edge
- Dual graph is plane.
- if G is connected ⇒ G* G

**Interdigitating Trees**

**Thm.** Let T be a spanning tree of planar graph G.

The dual N* of non-tree edges N form a spanning tree in G* (spanning tree of faces)

**Proof:**

**No cycle in N**

Assume there is cycle C

⇒ C divides plane into 2 parts

But T crosses C in at least one tree edge

Connected

T spans = |E(T)| = n - 1

⇒ |N| - |N*| = m - n + 1

want show |N*| more than ↑ by one.

by Euler ⇒ n - m + |N*| = 2 ⇒ |N*| = m - n + 2

![Diagram of a spanning tree](image)
Half-edge data structure ("chains", "bidirected" edge)

- in LEDA, OGDF

- Need often $O(1)$ queries for reporting
  - faces to left/right of an edge
  - end points of an edge
  - 1st edge incident to given vertex
  - clockwise/counter-clockwise successor edge in circular order (given $v, e$)
  - position of edge in incidence lists

\[
\text{face}(e_i)
\]

\[
\text{reverse}(e_i)
\]

- source
- target
- every face has representative edge.

Want:

Given 2 vertices, $v, w \in V(G)$ report whether they are adjacent.

**Constant Adjacency Queries**

$k$-orientation = orientation s.t. node has $\leq k$ outgoing edges.

**Static case:** By Euler's formula $\Rightarrow \exists$ vertex of degree $\geq 5$

- delete $v$, orient its edges out
- and keep doing (deletion preserve planarity)

\[
\Rightarrow \text{planar graph has acyclic } 5\text{-orientation}
\]

(If cyclic then 3-orientation)

So check outgoing edges of $v, w$ in $\mathcal{O}(1)$
Decremental Adjacency Queries

can delete & contract
easy! in O(1) → assume resulting graph is simple.

Want 14-orientation; if contraction exceeds 14 outgoing edges on \( w \)
put \( w \) in list \( L \)

While \( L \neq \emptyset \)
remove \( w \) from \( L \)
For each cut edge \( w \rightarrow v \)
add \( v \rightarrow w \) to outgoing \( [v] \)

\[ \text{reverse direction of } w \text{'s out edges.} \]
if outgoing \( [v] \geq 14 \) → add \( v \) to \( L \)

set outgoing \( [w] = \emptyset \)

Lemma: For any orientation \( O \) of planar graph \( G \), and \( v \in V \)
exists path from \( v \) to a vertex with outdegree \( \leq 3 \) in \( O \) of length \( \log n \)-1

Proof:
\( L_i \) = all nodes from \( v \), reachable by path of length \( i \)

Assume: \( L[\log n]-1 \) contain no outdegree \( \leq 3 \) node.

Will prove \( |L_{i+1}| \geq 2 |L_i| \)
\[ \Rightarrow |L_{i+1}| \geq 2 \log n \geq n^{1 \over 2} \]

Each vertex in \( L_i \) has outdegree \( \geq 4 \) \( \Rightarrow \) sum of outdegree \( \geq 4 |L_i| \)
but \( G[L_i] \) is plane \( \Rightarrow \) has \( \leq 3 |L_i| - 6 \) edges in \( L_i \)
\[ \Rightarrow \text{so } 3 \text{ edges leaving } L_i \text{ at least } |L_i| + 6 \]
\[ \Rightarrow |L_{i+1}| \geq 2 |L_i| \quad \blacksquare \]