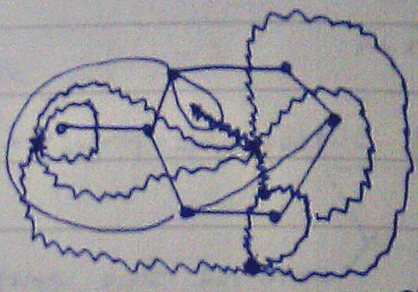
(Geometric) dual

Def.

G be a planar emb.

The dual graph G^* of G consists of

- a vertex f^* for each face
- a edge $e^* = f_1^* f_2^* \in E(G^*)$ for all f_1^*, f_2^* sharing one edge.



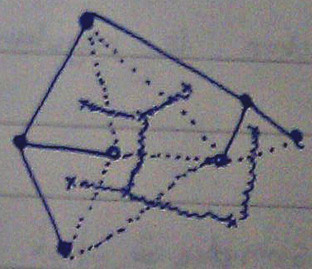
$E(G) \rightleftarrows E(G^*)$

- $\exists$ bijection edge $\leftrightarrow$ dual edge
- Dual graph is plane.
- if G is connected $\Rightarrow G^{**} = G$

Interdigitating Trees

Thm. Let T be spanning tree of planar graph G.
The dual N^* of non-tree edges N
form a spanning tree in G^* (spanning tree of faces)

Pf.)



No cycle in N^*

Assume there is cycle C

$\Rightarrow$ C divides plane into 2 parts

But $\Rightarrow$ T crosses C in at least one tree edge ↯

so can not be cycle.

Connected

T spans $\Rightarrow |E(T)| = n-1$

$\Rightarrow |N| = |N^*| = m-n+1$

want show $|V^*|$ more than $\uparrow$ by one.

by Euler $\Rightarrow$ $n - m + |V^*| = 2 \Rightarrow |V^*| = m-n+2$ ∎

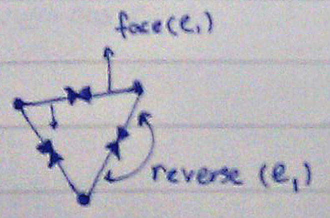
spanning tree!

Half-edge data structure ("darts", "bidirected" edge)

- in LEDA, OGDF

- Need often $O(1)$ queries for reporting
  - faces to left/right of an edge
  - end points of an edge
  - 1^{st} edge incident to given vertex
  - clockwise/ccw successor edge in circular order (given v, e)
  - " " a face
  - position of edge in incidence lists

old



- source(e1)
- target(e1)
- every face has representative edge.

$O(1)$ insert / delete edges & isolated nodes

Want

Given 2 vertices, $v, w \in V(G)$ report whether they are adjacent.

Constant Adjacency Queries

k-orientation = orientation s.t. node has $\leq k$ outgoing edges.

static case: by Euler's formular $\Rightarrow$
- $\exists$ vertex v of deg ≤ 5
- delete v, orient its edges out.

and keep doing (deletion preserve planarity)

$\Rightarrow$ planar graph has acyclic 5-orientation

(if cyclic then 3-orientation) note. we have

So check outgoing edges of v, w in $O(1)$

Decremental Adjacency Queries

can delete & contract

↓ easy! in $O(1)$ ↓ assume resulting graph is simple.

Want 14 - orientation ; if contraction exceeds 14 outgoing edges on w

$\Rightarrow$ put w in list L

While $L \neq \emptyset$

remove w from L

For each out edge $w \to v$

add $v \to w$ to outgoing[v]

if outgoing[v] > 14 $\Rightarrow$ add v to L

set outgoing[w] = $\emptyset$

} reverse direction of w's out edges.

Lemma For any orientation O of planar graph G, and $v \in V$

$\exists$ path from v to a vertex with outdegree ≤ 3 in O of length $\leq \lceil \log n \rceil - 1$

Pf) L_i = all nodes from v, reachable by path of length i

Assume : $L_{\lceil \log n \rceil - 1}$ contain no outdegree ≤ 3 node.

Will prove $|L_{i+1}| > 2|L_i| \Rightarrow |L_i| > 2^i$, $|L_1| \geq 5$ ~~so $i < \lceil \log n \rceil$~~

$\Rightarrow$ so $L_{\lceil \log n \rceil} > 2^{\lceil \log n \rceil} \geq n$ ↯

↓

Each vertex in L_i has outdegree $\geq 4 \Rightarrow$ sum of outdegree $\geq 4|L_i|$

but $G[L_i]$ is plane $\Rightarrow$ has $\leq 3|L_i| - 6$ edges in L_i

$\Rightarrow$ so $\exists$ edges leaving L_i at least $|L_i| + 6$

$\Rightarrow |L_{i+1}| - |L_i| \geq |L_i| + 6$

$\Rightarrow |L_{i+1}| > 2|L_i|$ ∎ :)