NP-Hard problems: Vertex Cover \( G=(V,E) \) - \( k \)

Assume (under Exp. Time Hypothesis) to need \( C^n \) time, \( c>1 \) known.

2) If \( k \) is small \( 1.28^k \) known.

Def: A problem \( P \) is parameterised if \( P \subseteq \{ (x,k) \mid k=\log n \} \implies 2^{O(k)} \) for any \( f(k) \).

Equivalent \( P \) in time \( n^{O(k)} \) for any fixed \( k \).

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**Vertex Cover (VC)**

Given \( (G,k) \)

1. Pick edge \( uv \in E(G) \)
2. Try \((G-u,v,k-1)\)
   - If pos. answer \( |X|=k-L \)
     - Try \((G-v,u,k-2)\) same way
3. Else reject

\[ \text{height} = k \]
\[ \text{size} \leq 2^k \]
\[ (G,x,0) \text{ Accept or Reject} \]

**Feedback Vertex**

Find \( k \)-vertices that hit every cycle \( c \) of \( G \).

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Interactive Compression.

Will show \( O^*(5^k) \) time alg.
Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$

Let bars denote vertices from $1$ to $n$ to indicate graph by taking first $i$ vertices
$G_i = G_{\{v_1, \ldots, v_i\}}$

Observe: FVS for $G_i$ is FVS for $G_i$ for all $i = n$
Feedback vertex set

Assume we know FVS $X_c$ for $G_i$, $|X_c| = k$

Then $X_c + v_{i+1}$ is FVS for $G_{i+1}$

Idea: Compress $X_c + v_{i+1}$ to optimal solution $|X_{i+1}| \leq k$ in $F(k)$ time

Gives FPT alg:

Input: $G, X, k$
$X$ is FVS, $|X| \leq k + 1$

1) for every $v \in X$ replace set
   1.1) if $G[\bar{v}]$ is not a forest (contains cycle) choose other $v$

12) call subroutine

Input II $(G, v_1, v_2, k')$
$G[\bar{v_1}], G[\bar{v_2}]$ forests

For FVS for $G$ contained in $v_1$:
1) delete leaves $d_G(v) = 1 = d_{\bar{v}}(v)$
2) $d_G(v) = 2$, $v \in v_1$, bypass $v$
   except if $N(v) \leq v_2$

$N(v) = \{w, w'\}$
Add edge $uw$, remove $v$, bypass $v$
Keep parallel edges
3) Pick \( v \in V_1 \) with \( \geq 2 \) ms \( u, w \in V_2 \)

3.1 If \( V_2 + V \) has cycle, delete \( v \)

3.2 Branch (b) move \( v \) from \( V \) to \( V \) 

Then \( Alg 2 \) runs in time \( O^*(4^k) \)

(assuming \( G \in V_1, G \in V_1 \) are forests, \( |V_2| \leq k + 1 \) and is correct)

Proof: Let \( v \) be a leaf in \( G \in V_1 \). If \( |N(v) \cap V_2| < 2 \) then \( d_0(v) = 2 \) and

steps 1 or 2 apply. Thus \( v \) exits for step 3.

In \( \# \) sol. 5, either \( V \in S \) or \( V \notin S \) this branch will find sol. 5.

Running time:

- Other branch (not sol)
  - More letter from \( V \) to \( V \)
  - \( \leq \# \) of conn. comp. \( V_2 \)
  - \( \leq k + 1 \) (\( V_2 \) is subset of prev. sol.)

\( \# \) of conn. comp. in \( G[V_2] \) = 2

\( \Rightarrow \) gives branching tree, height \( \leq p \leq k + (k+1) \leftrightarrow \) size \( O^*(4^k) \leftrightarrow O^*(4^k) \)

\( 2^p \leq 2^{2(k+1)} \leq 2^{2k} \)
Total work
\[ \sum_{S=x} y_{k-15} \]

\[ \sum_{i=1}^{\frac{k}{2}} (y_{k-i})^2 = \sum_{i=0}^{\frac{k}{2}} i \cdot (y_{k-i})^2 \]

\[ \sum_{i=0}^{\frac{k}{2}} \sum_{j=0}^{i} (y_{k-i})^2 = \frac{1}{4} \sum_{j=0}^{k/2} (y_{k-j})^2 + \frac{1}{4} \sum_{j=0}^{k/2} (y_{k-j})^2 = \frac{1}{4} \sum_{j=0}^{k} (y_{k-j})^2 = 0 \text{ (for } k) \]

(S: deleted part of \( x_y + v_{y+1} \))

Call \( H_y \) with \( k' = k - 15 \)

\[ y_{k-1} - y_{k-1} \cdot (k_{15}) - y_{k-1} \cdot (k_{16}) \]