NP-Hard problems / Vertex Cover $G=(V,E)$ - k

Assume (under Exp. Time Hypoth) to need c^n time, $c>1$ known

If k is small 1.28^k known.

Def A problem P is parameterized if $P \subseteq$ (ex. (G,k))

It is Fixed Param. Tractable (FPT) (F,k)

if $F(k)\,\mathrm{poly}(n)$ for some $f(k)$ ex $2^k n^2$

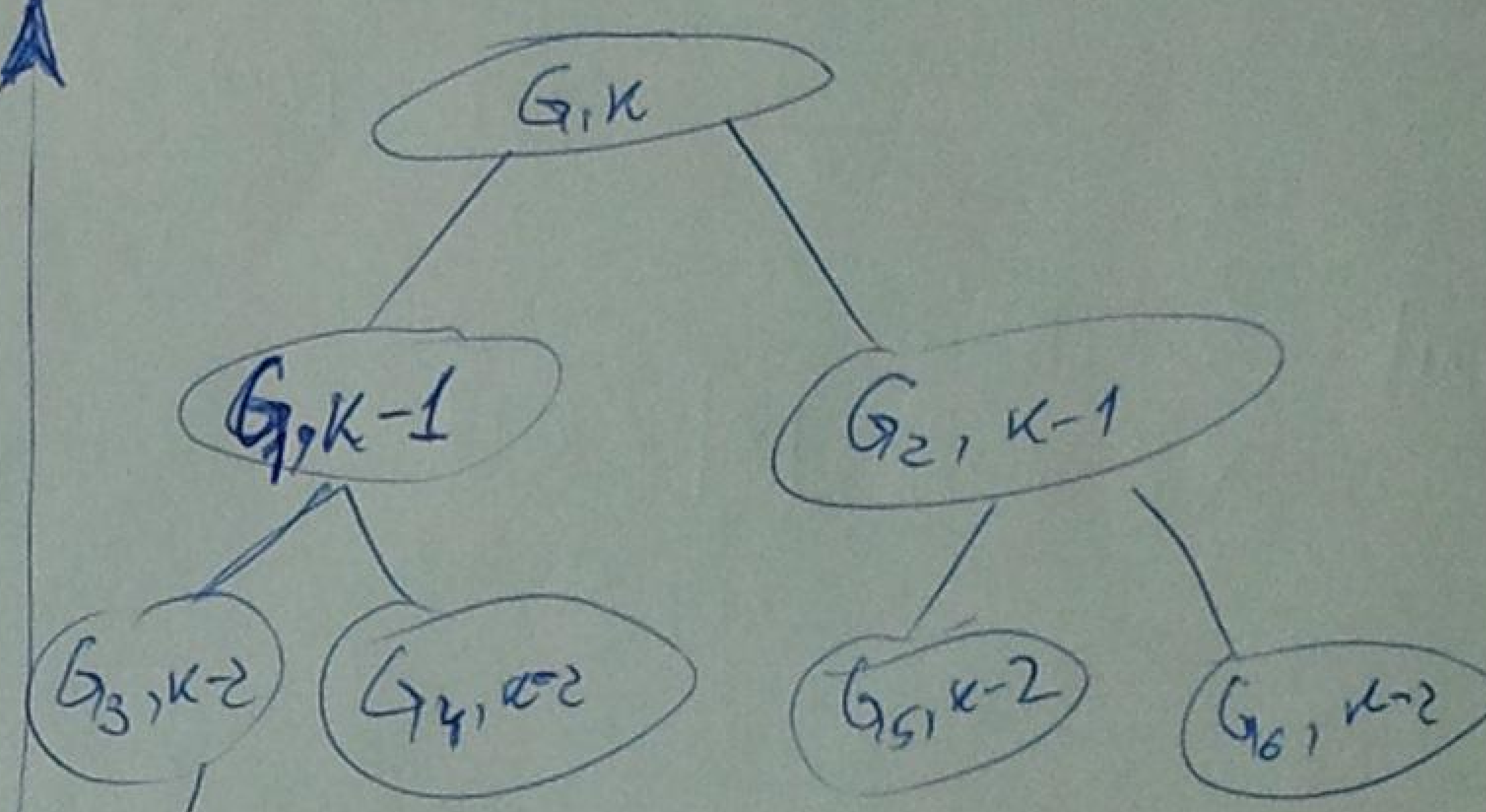
Equivalent P in time $n^{O(1)}$ for every fixed k

$k = \log n \Rightarrow 2^{\log n} \cdot f(k) = n \cdot f(k)$

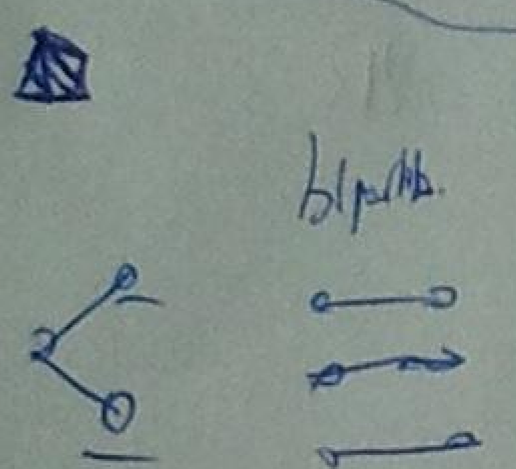
Ex: Trivial alg. VC: $n^{k+O(1)}$ enum ∀ subsets

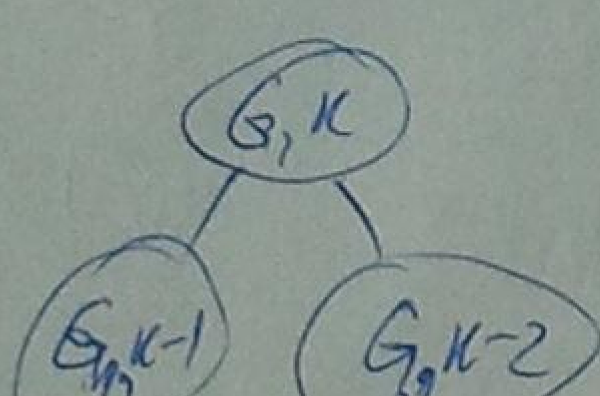
VC

Given (G,k)
1) Pick edge $uv \in E(G)$
2) Try $(G-u, k-1)$ $|X| \le k-1$
   If pos. answ $|X+u| \le k$
   Try $(G-v, k-1)$ same way
3) Else reject



height $\le k$
size $\le 2^k$
$(G_x, 0)$ Accept or Reject

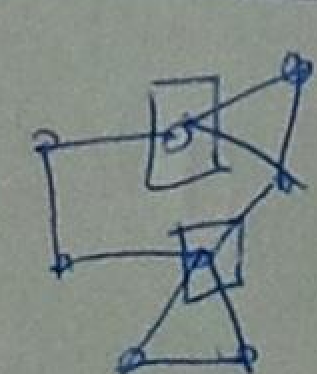




$T(k) = T(k-1) + T(k-2)$
$\Rightarrow$ fib. seq. $\Rightarrow \phi^k \approx 1.6181^k$

Feedback Vertex

Find k-vertices that hit every cycle C of G.



By simple branch & bound $\Rightarrow \infty$ n^k check ∀ subset $|H| = k$

$O^*(g) = O(g \cdot \mathrm{poly}\, n)$

Iterative Compression

Will show $O^*(5^k)$ time alg.

Let $G=(V,E)$, $V=\{v_1,\dots v_n\}$

Number vertices from 1 to n & induce graph by taking first i vertices

$G_i = G[\{v_1,\dots,v_i\}]$

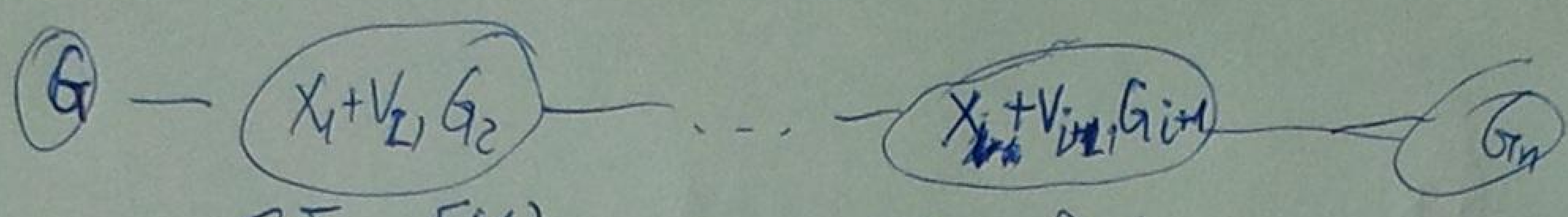
Observe: FVS for G is FVS for G_i for all $i \le n$ feedback vertex set

Assume we know FVS X_i for G_i, $|X_i| \le k$

Then $X_i + v_{i+1}$ is FVS for G_{i+1}

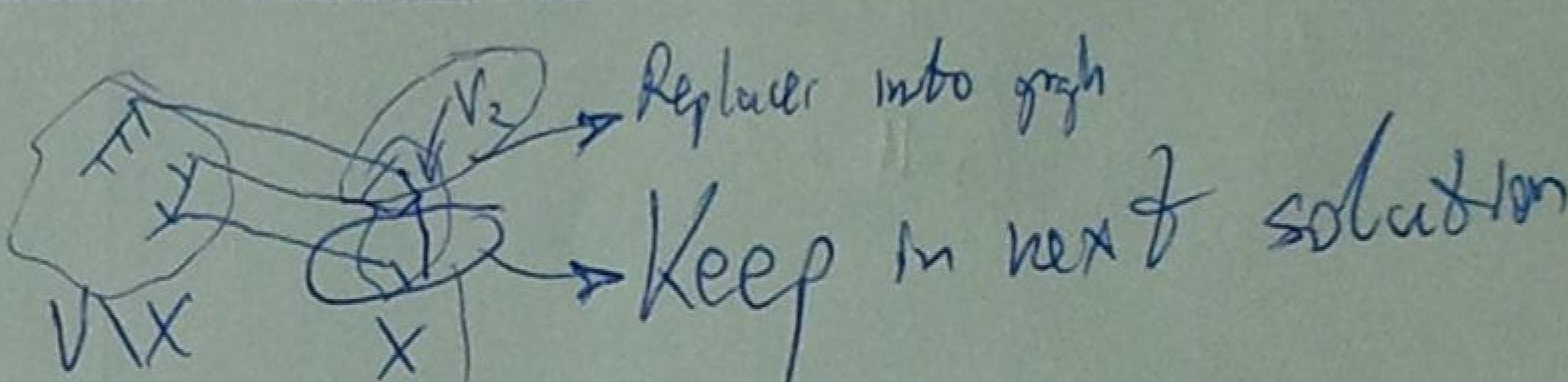
Idea: Compress $X_i + v_{i+1}$ to optimal solution $|X_{i+1}| \le k$ in $f(k)$ time

Gives FPT alg:



Input: G, X, k

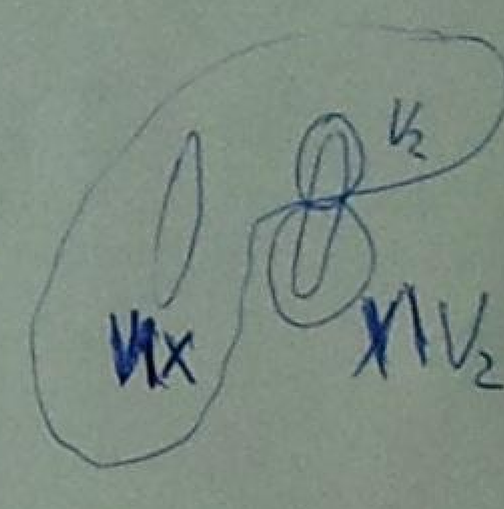
X is FVS, $|X| \le k+1$



$X\setminus V_2$ 2^{k+1} options

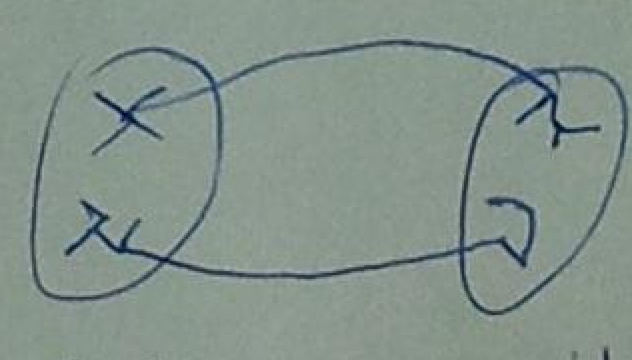
1) For every $V_2 \subseteq X$ Replaced set

1.1) If $G[V_2]$ is not a forest (contains cycle) choose other V_2 } Know we've made a wrong guessing



1.2) Call subroutine
$(G, V\setminus X, V_2, k-|X\setminus V_2|)$

$V\setminus X$ - forest since by assumption X is FVS $\Rightarrow$ $V\setminus X$ does not contain cycles!

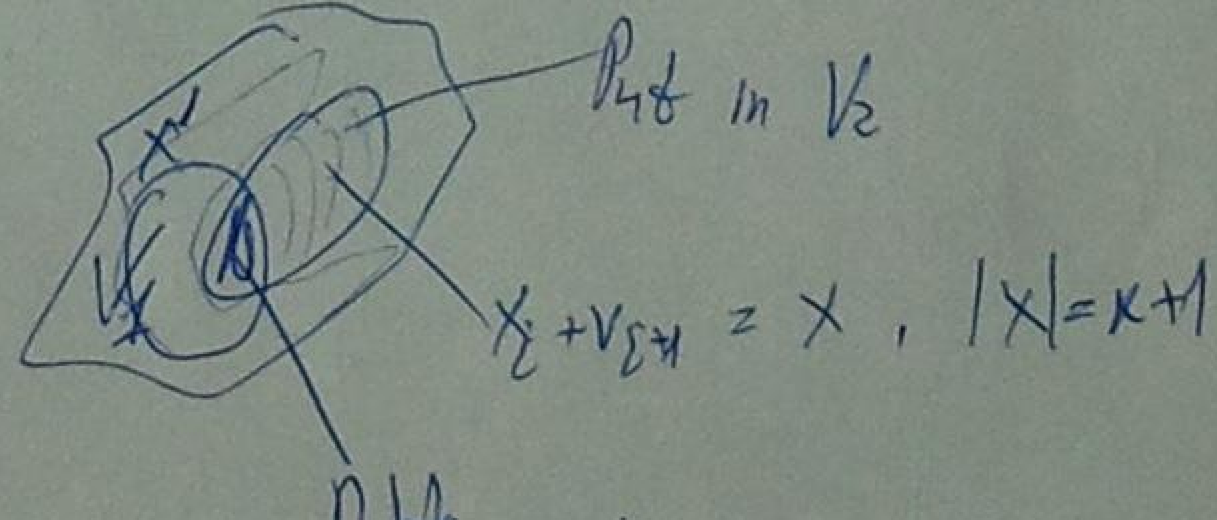


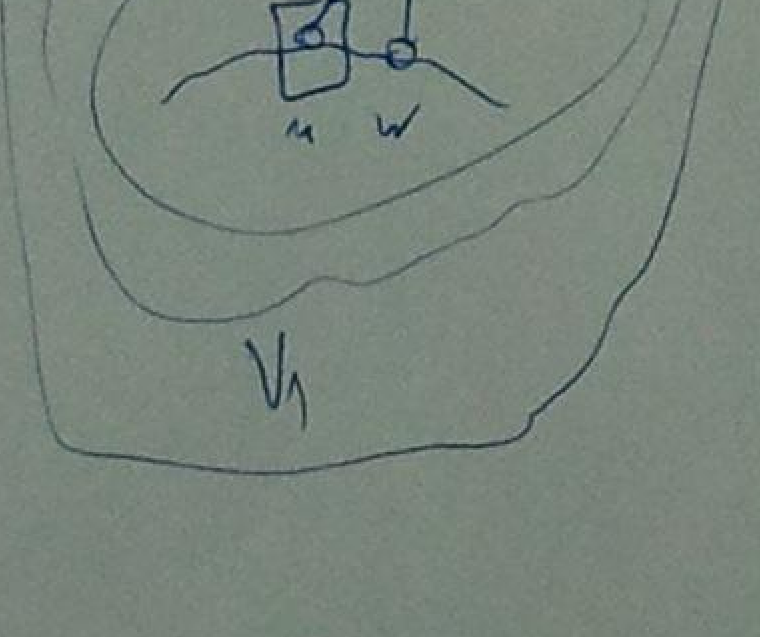
$V_1 = V\setminus X$ $|V_2| \le k+1$

Input II (G, V_1, V_2, k') $G[V_1], G[V_2]$ forests

For FVS for G contained in V_1

1. Delete leaves $\deg_G(v) = 1 = d_G(v)$
2. $d_G(v)=2$, $v\in V_1$, bypass v
except if $N(v) \subseteq V_2$

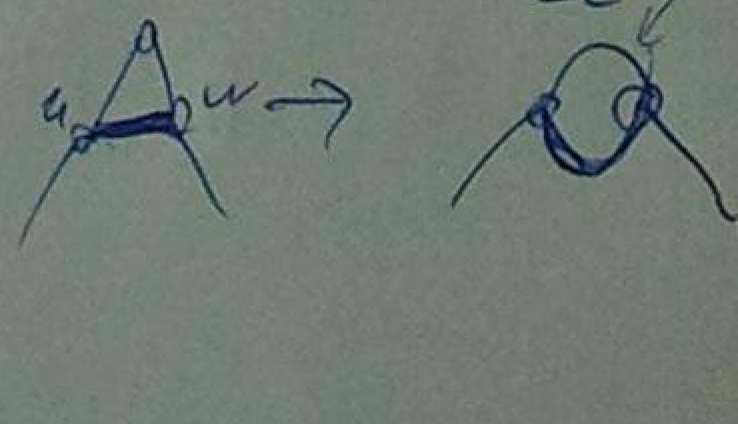




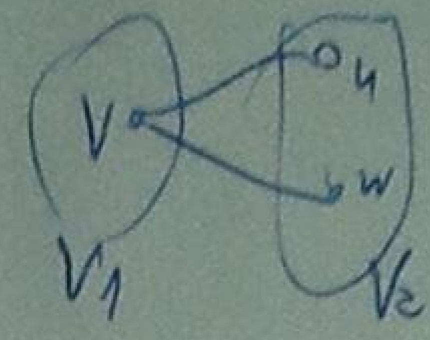
$N(v) = \{u,w\}$

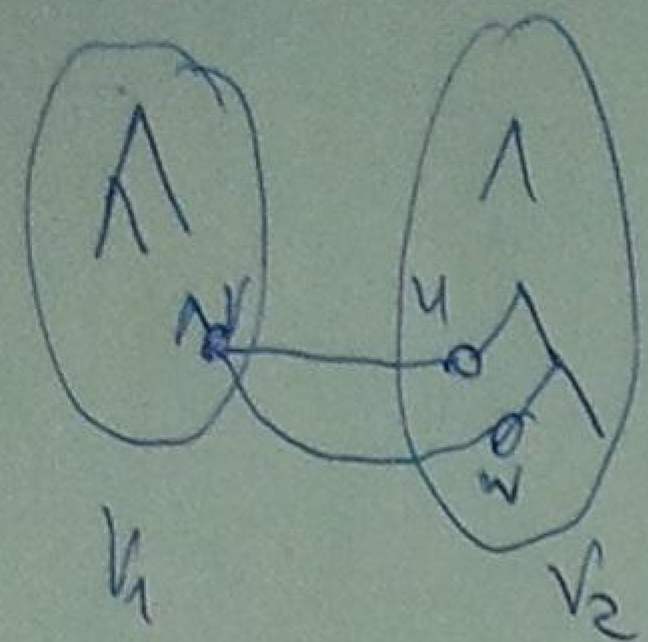
Add edge uw, remove v bypass

keep parallel edges



3) Pick $v \in V_1$ with ≥ 2 nbs $u, w \in V_2$





not allow to delete any vertex on V_2

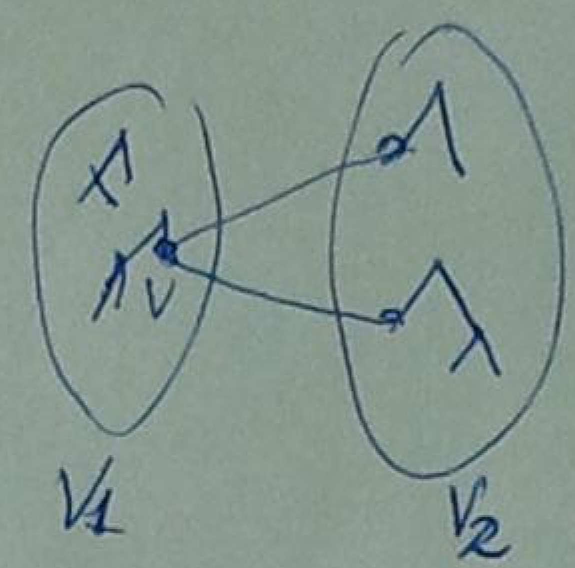
<u>3.1</u> If $V_2 + v$ has cycle, delete v

a) kill v & $k = k - 1$ & <u>recurse</u>!

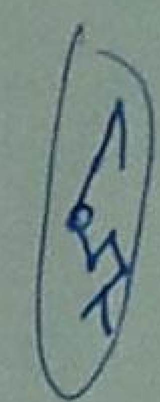
3.2. Branch (b) move v from V_1 to V_2 recurse

3.1 $(G \setminus v, V_1 \setminus v, V_2, k-1)$

3.2 $(G, V_1 - v, V_2 + v, k)$



$\Rightarrow$



V_2' # of trees in $G[V_2]$ decreases

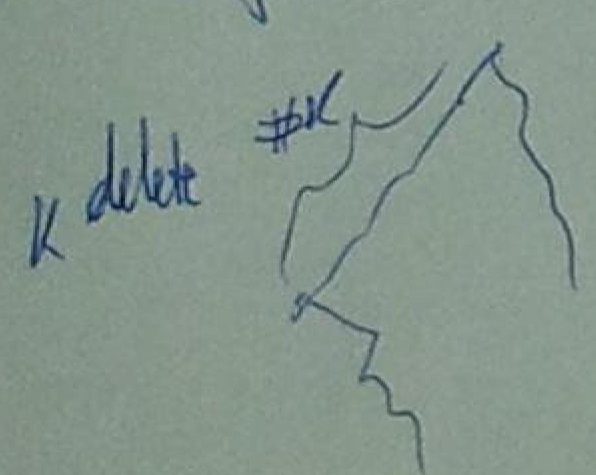
Thm: Alg 2 runs in time $O^*(4^k)$

(assuming $G[V_1], G[V_2]$ are forests, $|V_2| \leq k+1$ & is correct)

Proof: Let v leaf in $G[V_1]$. If $|N(v) \cap V_2| < 2$ then $d_G(v) \leq 2$ and steps 1 or 2 apply. Thus $\exists$ vertex for step 3.

In $\forall$ sol. S, either $v \in S$ or $v \notin S$ this branch will find sol. S.

<u>Running time</u>:



other branch (not del)

Move vertex from V_1 to V_2

$\leq$ # of con. com. of V_2

$\leq k+1$ (V_2 is subset of prev. sol.)

Let $p = k + l$ conn. comp of $G[V_2]$ then in every branch p decrease.

$$\begin{array}{ccc} & k, l & \\ \text{kill} \ \swarrow & & \searrow \\ k-1, l & & k, l-1 \end{array} \qquad k, l \geq 0$$

# of conn. comp. in $G[V_2] = l$

$\Rightarrow$ Gives branching tree, height $\leq p \leq k + (k+1)$ $\Rightarrow$ size $O^*(4^k)$ $\Rightarrow$ $O^*(4^k)$

$2^p \leq 2^{2k+1} \leq 2 \cdot 4^k$

Total work

$$\sum_{S \subseteq X} 4^{k-|S|}$$

(S: deleted part of $X_i + v_{i+1}$
call Alg. with $k' = k - |S|$

$$= \sum_{i=0}^{|X|=k+1} \binom{|X|}{i} 4^{k-i}$$

$$4^{k-i} = \frac{1}{4} 4^{(k+1)-i}$$

$$= \frac{1}{4} \sum_{i=0}^{k+1} \binom{k+1}{i} 4^{(k+1)-i} \cdot 1^i = \frac{1}{4}(4+1)^{k+1} = O(5^k)$$