Kernelization

A polynomial time reduction $(G,k) \to (G',k')$ s.t.
and $|G'| \le f(k)$, $k' \le k$
and (G,k) pos. inst. iff (G',k') pos.

$f(k)$ poly $\Rightarrow$ polynomial kernelization

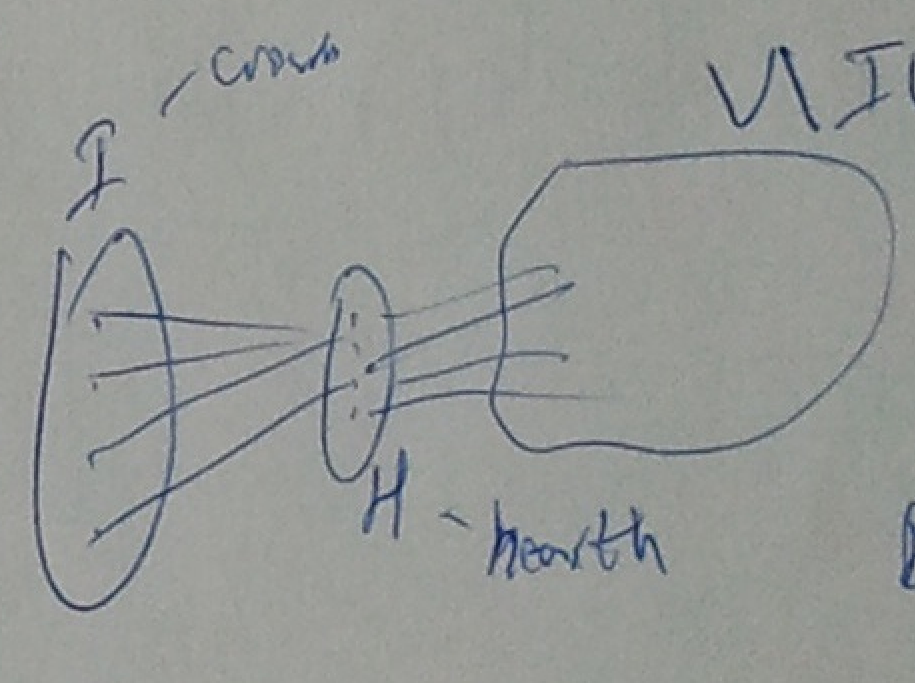
Reduction Rule:

R1: If $d(v)=0$ discard v : $(G,k) \to (G-v, k)$
R2: If $d(v)>k$, pick v : $(G,k) \to (\underline{G-v}, \underline{k-1})$
R3: If R2 does not apply but $|E| > k^2$, reject the instance
$\forall v \in V: d(v) \le k$: $\le k|X|$ edges
$\Rightarrow$ VC has ~~size~~ $O(k^2)$ ~~edge~~ vertex kernel
Else $|E(G)| \le k^2$, $|V(G)| \le |E(G)| \le k^2$

Def: A crown in G
- I indep. set
- $H = N(I)$
- H matched by I in bipartite graph (I,H)

$|I| \ge |H|$

no edges within set I

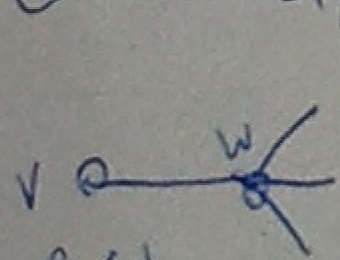


Claim: If (I,H) is crown in VC, delete H
do $(G,k) \to (G \setminus I \cup H, k - |H|)$

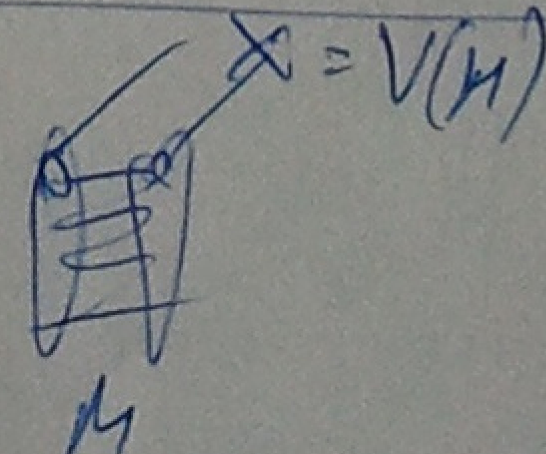
Any VC needs $\ge |H|$ vertices $G[I \cup H]$?
by matching

Formally S is VC for G, then $S' = (S \setminus I) \cup H$ is VC for G, $|S'| \le |S|$

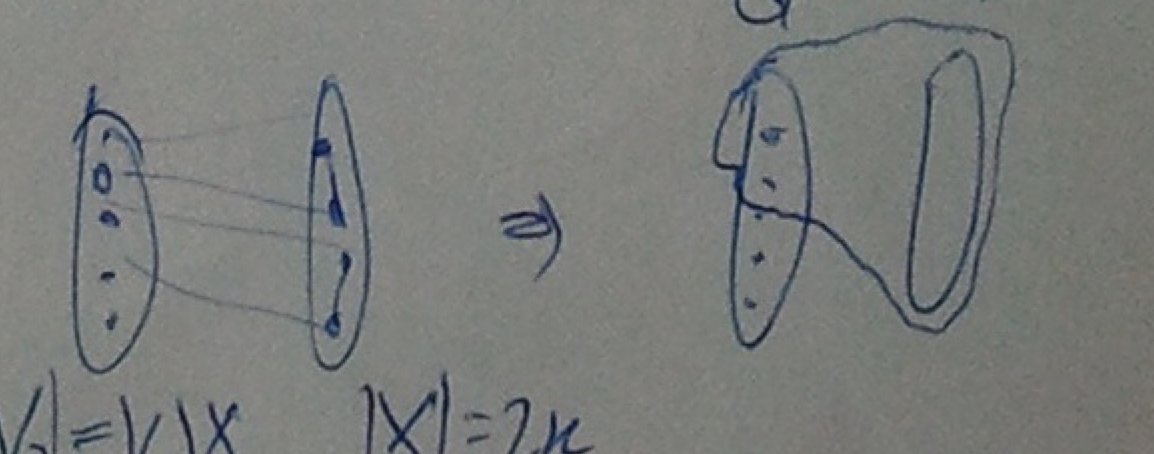
Ex. $d(v)=1$



Thm: VC has 3k-vertex kernel



Proof: Let M maximal matching, X endpoints of
$|M| > k \Rightarrow$ reject
Else, let $G' = (V \setminus X \cup X, E')$ be bipartite graph on



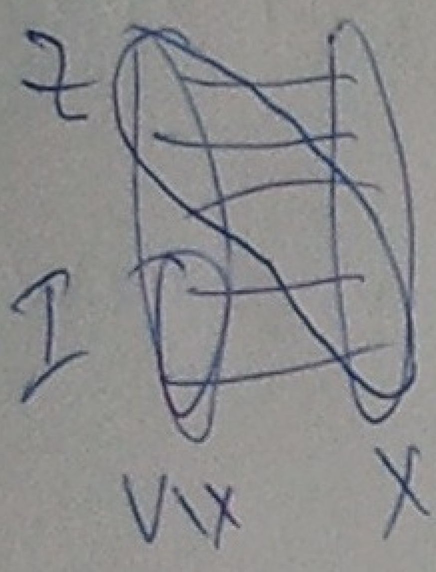
$Z = VC$ for G' $\quad$ $Z = \min VC$ for G'

If $|Z| > k$ reject $\quad$ size of $|Z| \leq k$

Case 1: $Z \cap X = \emptyset \Rightarrow Z = V_2$ so $|V_2| \leq k \Rightarrow |V| \leq k + 2k = 3k$

Case 2: $Z \cap X = H \neq \emptyset$

Recall: G' bipartite $|\text{Smallest } VC| = |\text{Maximal Matching}|$



$Z = \min VC$

$Z \cap X \neq \emptyset$

Z is $VC(G')$ let M' be matching G' of size $|Z|$

Observe $\forall \{u,v\} \in M'$: $|\{u,v\} \cap Z| = 1$

Let $H = Z \cap X$ head

$I = (V \setminus X) \setminus Z$

Want $I \subseteq V \setminus X$ s.t. $N(I)$ is matched by I

1) I indep. crown

2) $H = N(I)$ since Z is VC, $H =$

3) H covered by matching by I

Use M': M' goes between $(V \setminus X)$ and X, and between $(V \setminus Z)$ and Z

$\Rightarrow$ between I and H.

Conclude: Unless $n \leq 3k$ we can simplify $G \Rightarrow 3k$ vertex kernel.

FVS Kernel:

R1: IF $d(v) \le 1$ drop v : $(G, k) \to (G - v, k)$

R2: IF $e = uv$ is a bridge, remove it

R3: If $d(u) = 2$ bypass u (allow double edges

(a) u v $\Rightarrow (G - u - v, k - 1)$ Del. v

(b) $u <^{v}_{w}$ $\Rightarrow$ v | w, $k' = k$

(c) $u <^{v}_{w}$ $\Rightarrow$ v () w $k' = k$

(d) $u <^{v}_{w}$ $\Rightarrow$ v () w $k' = k$

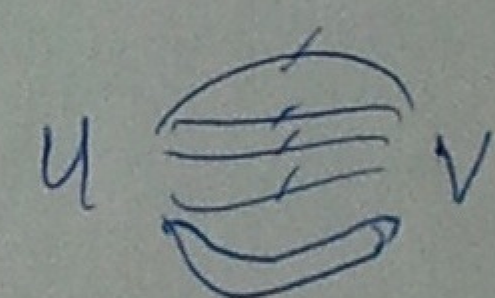
Reason.
Exists optimal FVS w/o u
$\Rightarrow S - u + v$, $S - u + w$

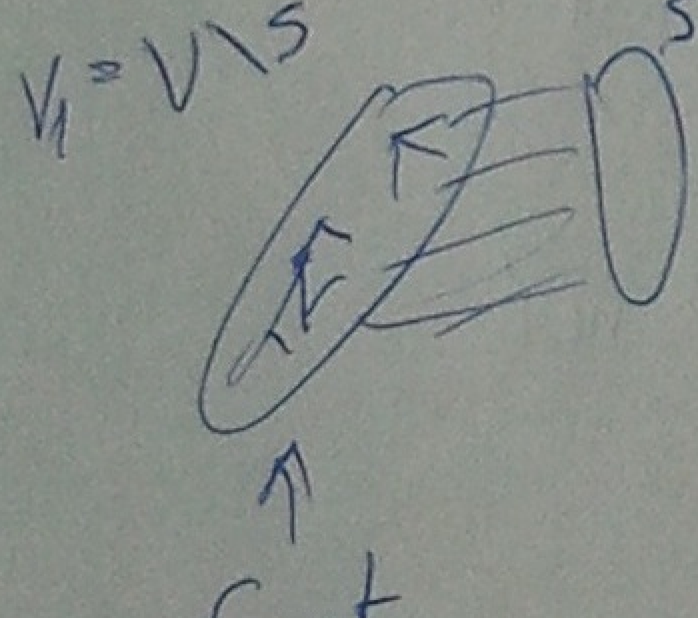
2-cycle $v \subset w$ is a cycle will be hit by FVS

$\boxed{\text{Now } \underline{d(v) \ge 3} \ \forall v \in V}$

pairs u v

Rule 4: IF $uv \notin E$ but $(k+2)$ v-disjoint paths $u - v$
add double edges $u \rightleftharpoons v$





$V_1 = V \setminus S$

S Unknown FVS $|S| \le k$

forest S by assumption

$V_1 = V \setminus S$ is $G[V_1]$ forest

$m(G[V_1]) \le |V_1| - 1$

but $\sum_{v \in V_1} d_G(v) \ge 3|V_1|$

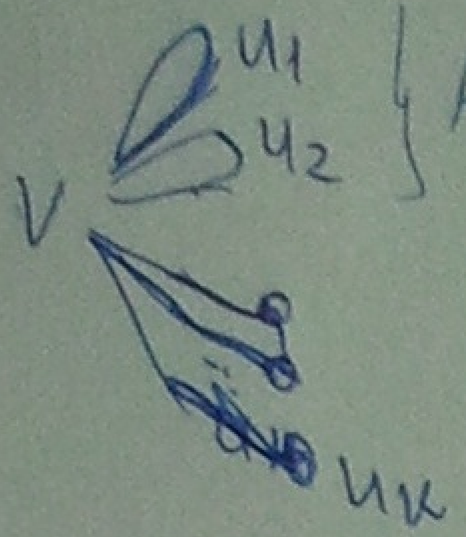
#edges from V_1 to S $\longrightarrow$ $\ge 3|V_1| - 2(|V_1| - 1)$

$\ge |V_1| = n - k$ edges from V_1 to S

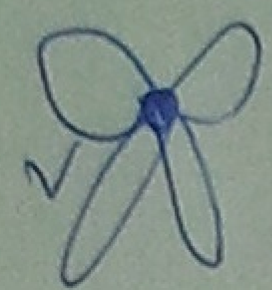
So $\exists v \in S$ with $d(v) \ge \frac{n-k}{k}$

Assume

Assume $n \geq 2k^3 + 8k^2$, so $d(v) \geq 2k^2 + 8k$ exists
else reject
let v be s.t.



"Flower"

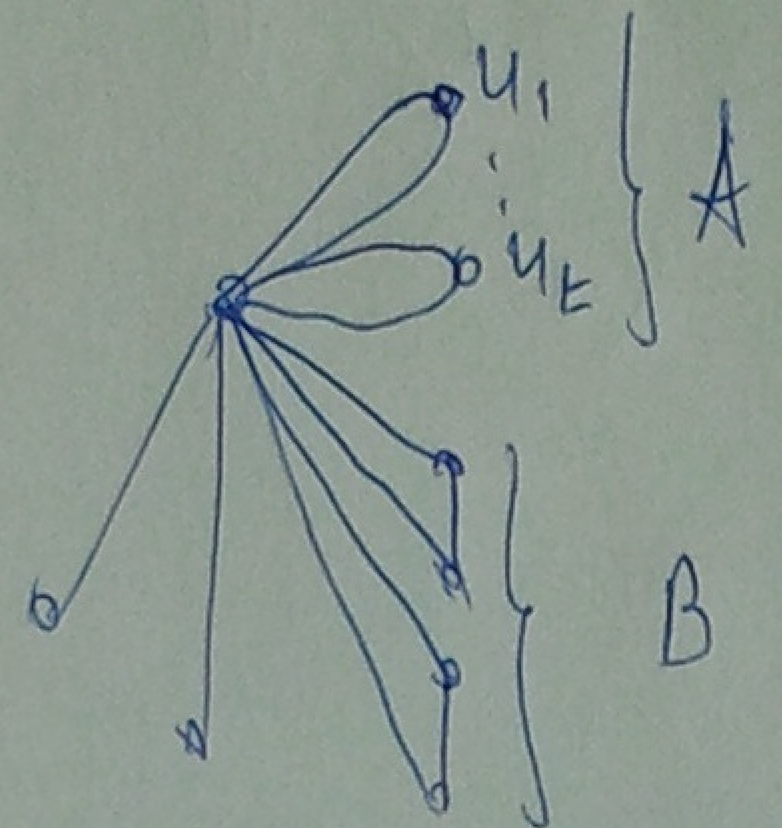


$k+1$ cycles w/ all $\cap$ in vertex v.

Goal:

Rule: Delete v

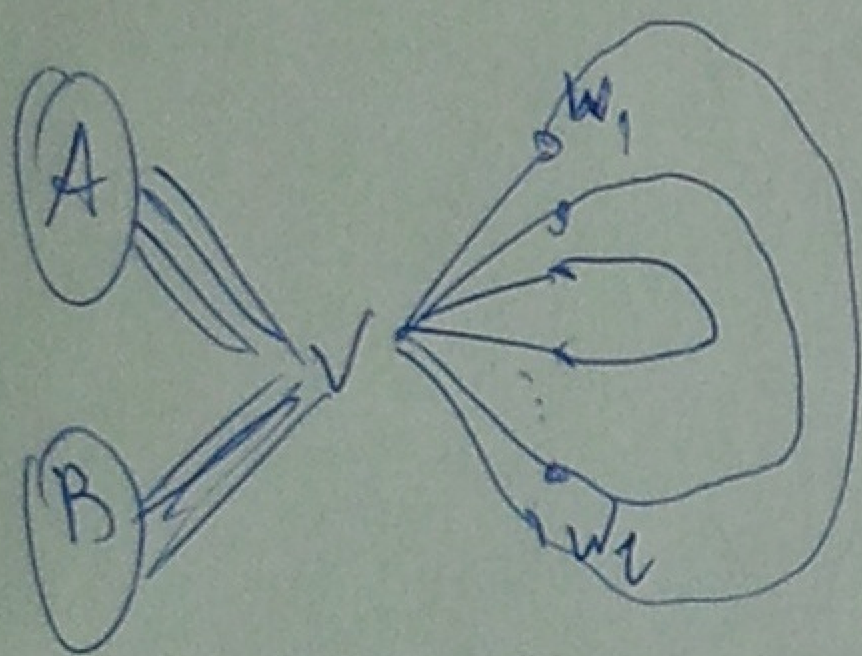
Matching in $N(v)$ ~~st~~



If size of $|A| + \frac{|B|}{2} > k$
have flower.

Else

$C := N(v) \setminus (A \cup B)$, $|C| \geq 2k^2 + 8k$



Path u-w a path which avoid v

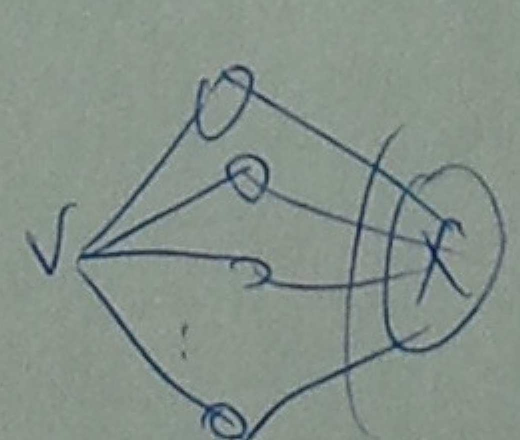
$v = u$, $w \in C$

Block ~~box~~ Lemma BBL

Can find either set $k+1$ v C path ~~v-W~~. 2.#paths
or set X, $|X| \leq 2k$, which hits ~~A~~ such path

Will show in case 2, contradict rule 4 ("Flower")

Consider $G \setminus (A \cup B \cup X)$ conn comp.

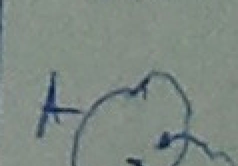


Exists some cycle
with each edge

$\Rightarrow$ each conn. comp. C_i is NB of X

$G = (V, E)$
$A \subseteq V$ be set terminals

Def A-path – path with endpoints in A and internal vertices not in it



Set of A-path = vertex disjoint A-paths

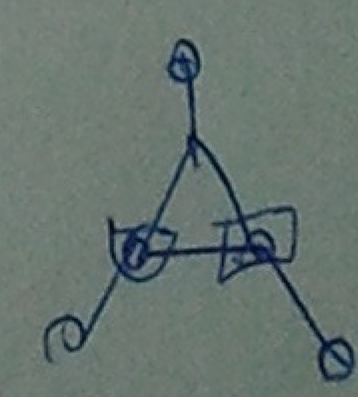
Max # of such paths is ≤ 2 size of min A cut

~~Set X st~~

Get $|C| \geq 2k^2 + 8k = 2k(k+4)$ path from v to X, $|X| \leq 2k$ by BBL

$\Rightarrow$ ~~Exists~~ $x \in X$: Flow $(v-x) > k+2$

x incident to
at least $k+2$ diff. comp.



A path

A cut of size 2

1 Addendum to "cubic kernel" for Feedback Vertex Set

As shown by Stéphan Thomassé, the Feedback Vertex Set problem has a quadratic kernel [1]. (Thomassé gives the result as $4k^2$ vertices; a careful analysis shows that in fact, the kernel also has total size $\mathcal{O}(k^2)$, i.e., with $\mathcal{O}(k^2)$ edges.)

In the lecture, I tried to give a shorter, simpler proof of a weaker cubic bound (i.e., with $\mathcal{O}(k^3)$ vertices). Unfortunately, the sketch in the lecture is incomplete. After the set X is removed, in the last part of the proof, it is claimed that because every edge occurs in a cycle, every component neighbouring x must be connected to X. This is incorrect – every component neighbouring x must be connected to X or to $A \cup B$. In particular, there is so far nothing in the kernel to prevent a large number of components neighbouring only x and A.

Now, if C is a connected component in $G \setminus (A \cup \{x\})$, and connected to x with only a single edge, while every $v \in A$ is connected to x via a double edge, then it can be shown that the edge from x to C may be removed in G (because if it is part of a cycle, then the graph also contains a shorter cycle with x and some $v \in A$). This would give an extra **Reduction Rule**, removing edges from x to components whose only other neighbours are in A (i.e., double neighbours of x). The proof would then proceed as in the lecture, with every resulting component neighbouring B or X, and with a new double edge as a conclusion, giving a cubic kernel.

However, given that the problem already has a quadratic kernel (Thomassé [1]), the interest in completely formalizing this cubic variant seems limited. If you're curious, I encourage you to read Thomassé's paper.

Note also that the termination of the reduction rules has to be argued – we now have rules that both add and remove edges, without reducing n or k. But this is simple to show, as every rule either removes a *single* edge or introduces a *double* edge.

Magnus Wahlström, June 2012

References

[1] S. Thomassé. A $4k^2$ kernel for feedback vertex set. *ACM Transactions on Algorithms*, 6(2), 2010.