Maximum Weighted Matching

Ran Duan

In this lecture

- Hall’s theorem
- Maximum weighted bipartite matching
- Hungarian algorithm

Hall's Theorem

- Given a bipartite graph $G=(L \cup R,E)$, where $|L|=|R|$,
    - It contains a perfect matching if and only if:
    - For every subset $S \subseteq L$, $|\Gamma(S)| \geq |S|$
    - ($\Gamma(S)$ is the set of vertices adjacent to S)

Proof on Induction

- Let $n=|L|=|R|$
- When $n=1$, trivial
- Suppose it holds for all $n \le k$, for $n=k+1$, two cases:

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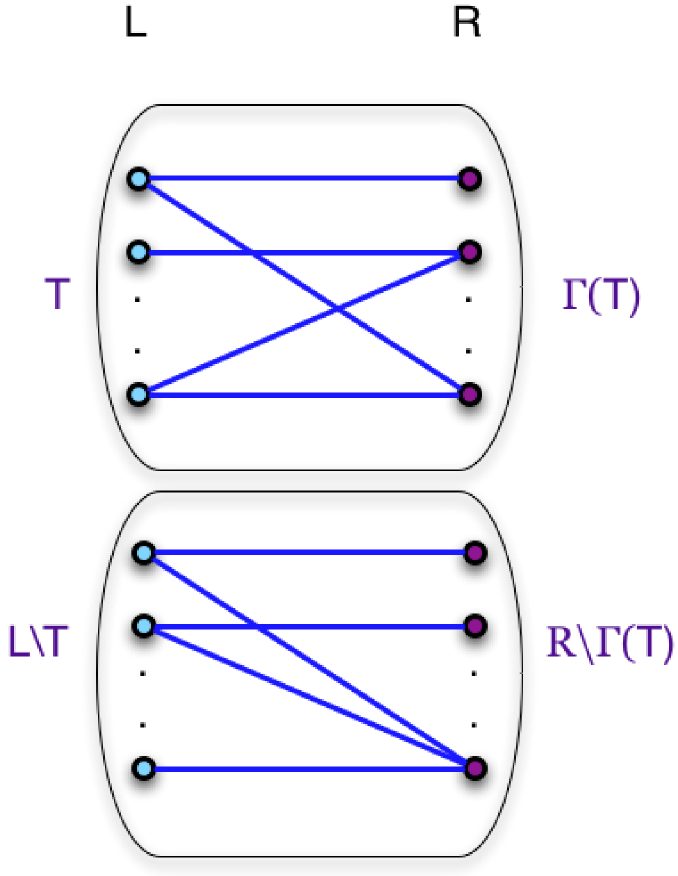
- Case I: For every subset $S \subseteq L$, $|\Gamma(S)| \ge |S|+1$
  - Then we arbitrarily put an edge (u,v) in the matching
  - In $G-\{u,v\}$, it still satisfies the condition $|\Gamma(S)| \ge |S|$, so the result holds by the induction condition

Proof on Induction

- Case II: there exists a $T \subseteq L$ which has $|\Gamma(T)| = |T|$, then the subgraphs of G on:
  - $T \cup \Gamma(T)$
  - $(L \setminus T) \cup (R \setminus \Gamma(T))$

  both satisfies the Hall's condition



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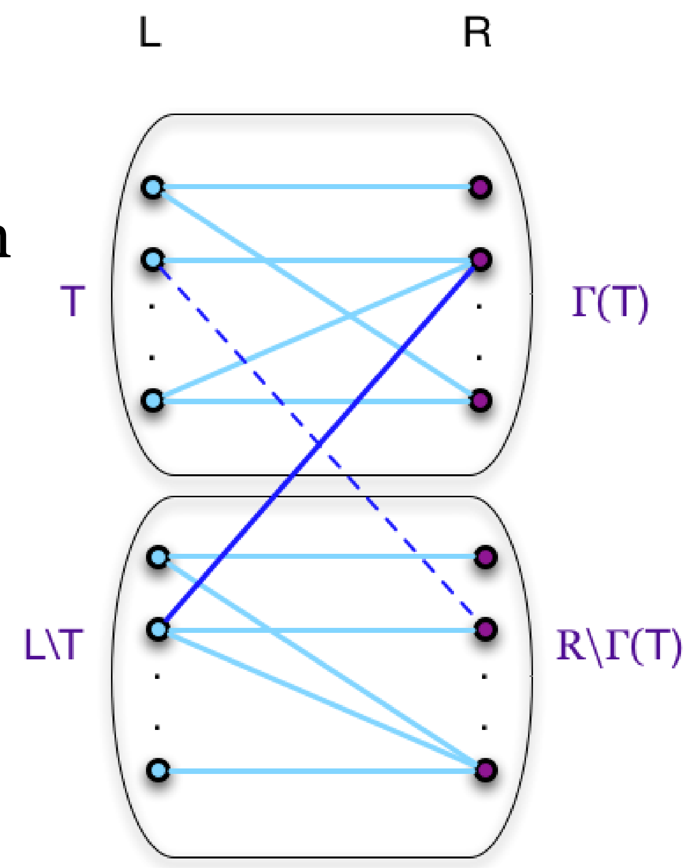
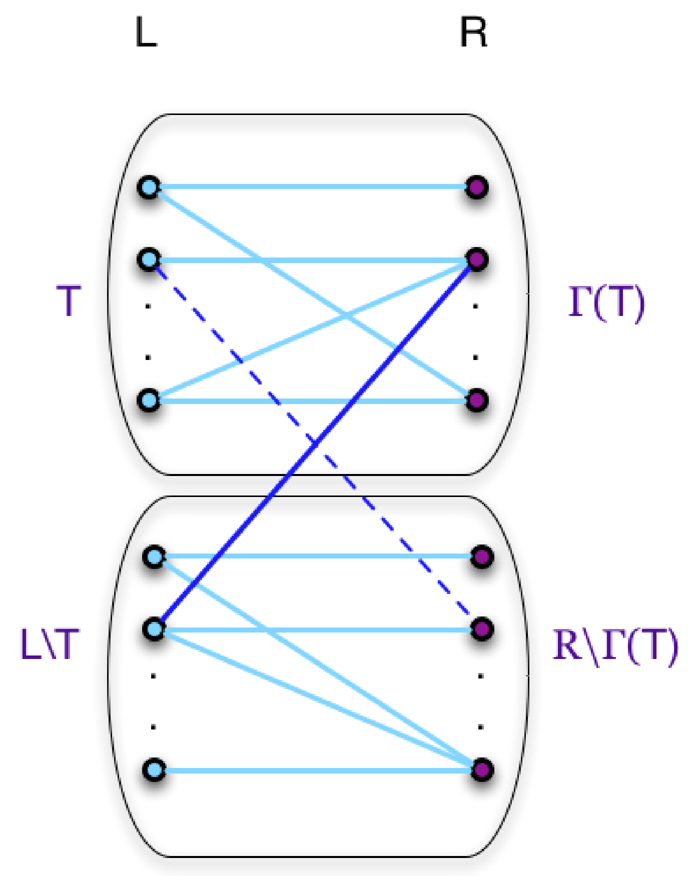


There may be an edge between L\T and Γ(T)
But there are no edge between T and R\Γ(T)

Proof on Induction

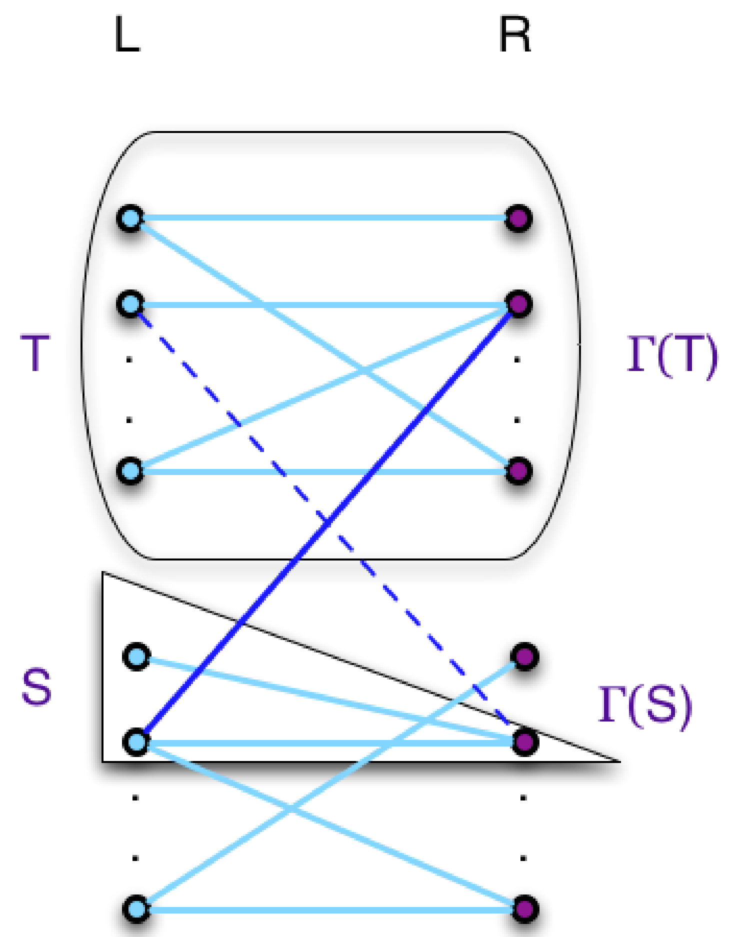
- In $T \cup \Gamma(T)$, every $S \subseteq T$ have $\Gamma(S) \subseteq \Gamma(T)$, so it satisfies the Hall's condiction



Proof on Induction

- In $(L\backslash T)\cup(R\backslash\Gamma(T))$,
if $\exists S\subseteq L\backslash T$ having
$|\Gamma(S)\cap(R\backslash\Gamma(T))|<|S|$,
then $T\cup S$ will also break the Hall's condition for G,
a contradiction

So $(L\backslash T)\cup(R\backslash\Gamma(T))$ satisfies the Hall's condition



Proof on Induction

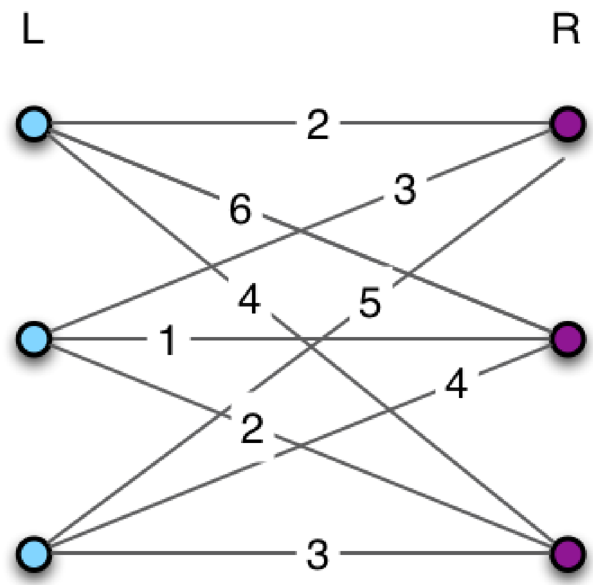
- Case II: there exists a $T \subseteq L$ which has $|\Gamma(T)| = |T|$, then the two subgraphs of G on:
  - $T \cup \Gamma(T)$
  - $(L \setminus T) \cup (R \setminus \Gamma(T))$

  both satisfies the Hall's condition

So we can find perfect matchings in these two subgraphs, and finally get a perfect matching of G.

Weighted Bipartite Matching

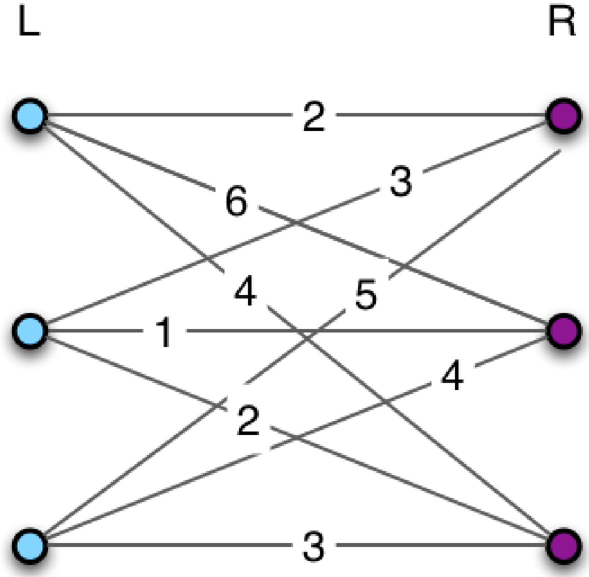
- Maximum Weighted Matching (MWM)
  - Maximize $\sum_{e \in M} w(e)$



Assignment Problem

- In operation research:
  - Some agents, some tasks
  - Assign each task to a agent
  - Maximize efficiency or minimize cost

	Cleaning	Sweeping	Washing
Jim	\$2	\$6	\$4
Steve	\$3	\$1	\$2
Alan	\$5	\$4	\$3

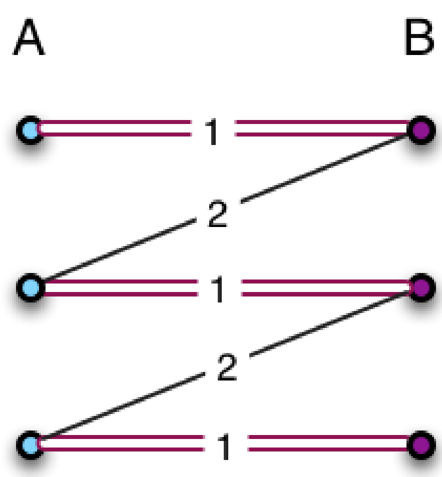


Weighted Bipartite Matching

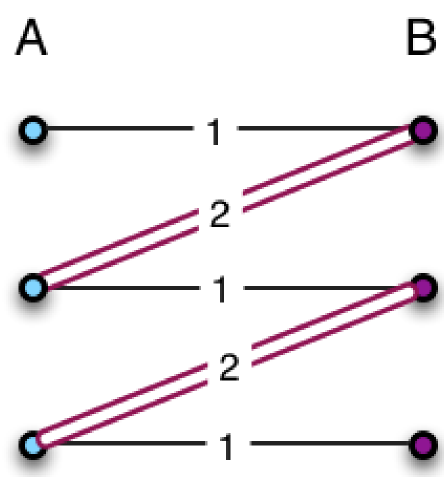
- When not every pair of vertices of L and R has an edge, we can consider two problems:
- Maximum (Minimum) perfect matching
  - The maximum or minimum among all perfect matchings
- Maximum matching
  - Not necessarily perfect

Weighted Bipartite Matching

- When not every pair of vertices of L and R has an edge, we can consider two problems:
- Maximum (Minimum) perfect matching (MWPM)
    - The maximum or minimum among all perfect matchings
- Maximum matching (MWM)
    - Not necessarily perfect



MWPM



MWM

Reduction between MWM and MWPM

- MWM=>MWPM
  - We add zero-weight edge for any pair of (u,v) if there is no edge between (u,v). ($u \in L$, $v \in R$)
  - In the new graph any matching can be extend to a perfect matching of the same weight, so the maximum perfect matching must have maximum weight.

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  - It will increase the number of edges

Reduction between MWM and MWPM

- MWM=>MWPM
  - Duplicate G, we have $G_1=(L_1,R_1)$ and $G_2=(L_2,R_2)$.
  - Link the two copies of every vertex of G by an edge with weight zero
  - Still a bipartite graph: one side $L_1 \cup R_2$, the other side $L_2 \cup R_1$

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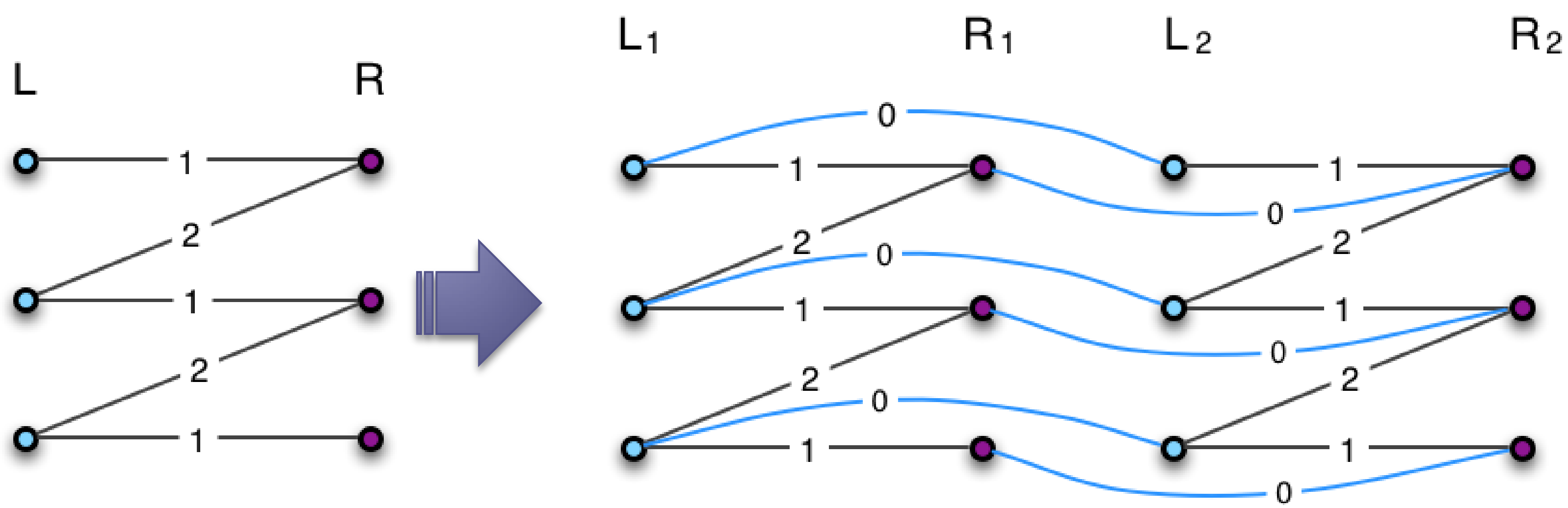

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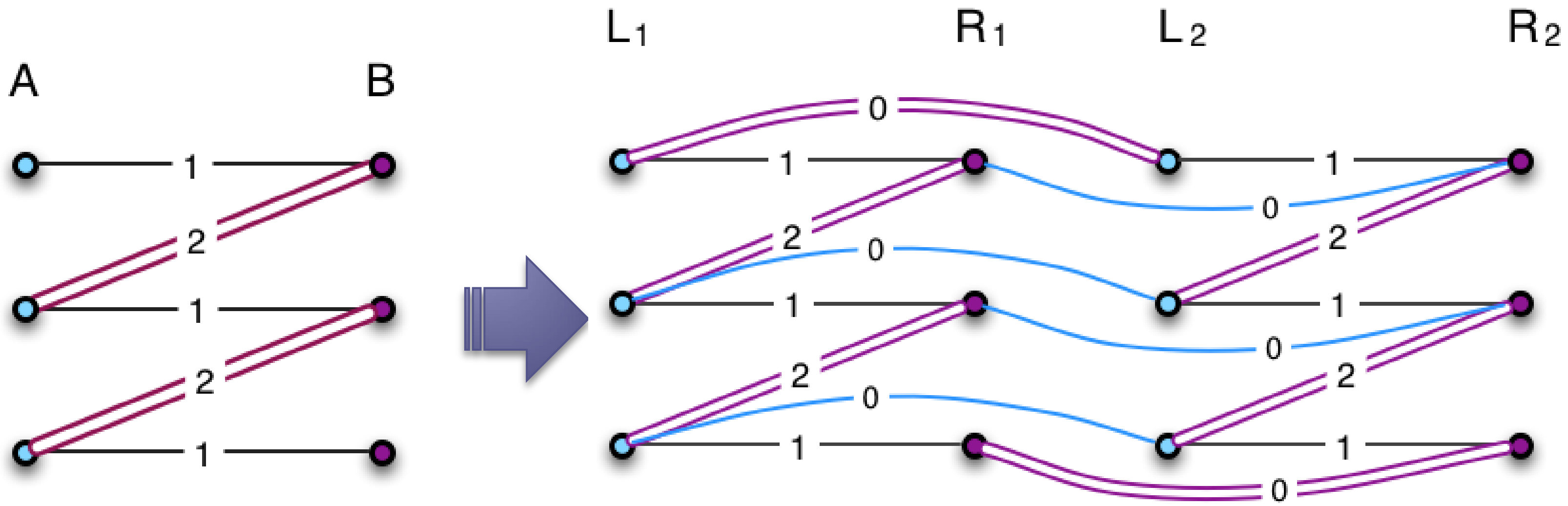

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  - Duplicate G, we have $G_1=(L_1,R_1)$ and $G2=(L_2,R_2)$.
  - Link the two copies of every vertex of G by an edge with weight zero
  - Still a bipartite graph: one side $L_1 \cup R_2$, the other side $L_2 \cup R_1$
  - The number of vertices and edges are still O(n) and O(m), respectively.

Reduction between MWM and MWPM

- MWPM=>MWM
  - If the weights are in $[0,\ldots,N]$, add nN to the weight of every edge, and get a new graph G'
  - The weight of a matching of k edges in G' is $\leq k(n+1)N$ (when $k \leq n-1$, $k(n+1)N < n^2N$)
  - The weight of a perfect matching in G' is $\geq n^2N$
  - So the maximum matching in G' must be a perfect matching.

Hungarian Algorithm

- By Harold Kuhn in 1955, who gave the name because it was largely based on the earlier works of two Hungarian mathematicians: Dénes Kőnig and Jenő Egerváry.
- In 2006, it was discovered that Carl Jacobi had solved the assignment problem in the 19th century.

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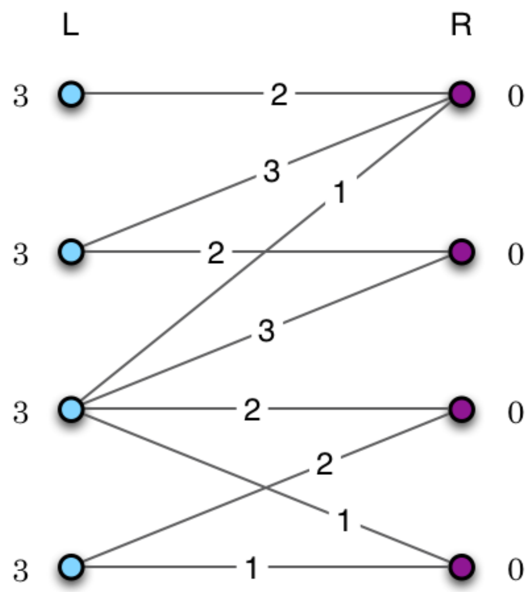
- We will first talk about the maximum perfect matching.

Hungarian Algorithm

- Dual variable y: L∪R→Z satisfies:
- For every e=(u,v), y(u)+y(v)≥w(e)



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- So for every perfect matching M,

$$w(M) = \sum_{e \in M} w(e) \leq \sum_{v \in L \cap R} y(v)$$

Hungarian Algorithm

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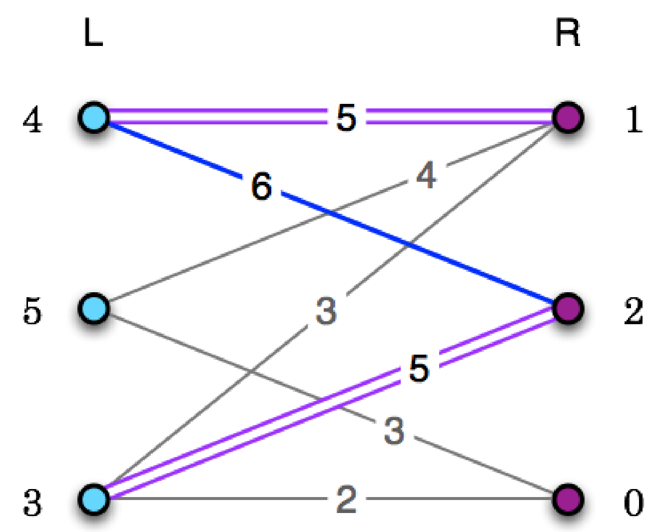
$$w(M) = \sum_{e \in M} w(e) \leq \sum_{v \in L \cap R} y(v)$$

- Our aim: obtain a perfect matching M* s.t.
  - for every $e \in M^*$, $y(u)+y(v)=w(e)$

- Throughout the algorithm:
  - $y(u)+y(v)\geq w(e) \qquad \forall\ e=(u,v) \qquad$ (domination)
  - $y(u)+y(v)=w(e) \qquad \text{if } e\in M \qquad$ (tightness)

- Throughout the algorithm:
  - $y(u)+y(v) \geq w(e)$ $\forall$ $e=(u,v)$ (domination)
  - $y(u)+y(v) = w(e)$ if $e \in M$ (tightness)
- Tight edges:
  - An edge $e=(u,v)$ is tight if $y(u)+y(v)=w(e)$
  - Denote the subgraph of tight edges by G_y



Procedure

- Let $y(u)=N$, $y(v)=0$ ($u \in L$, $v \in R$)
- Repeat
  - Augment M in G_y (subgraph of tight edges), until there is no augmenting path any more.
  - If M is not perfect, do the dual adjustment to make more edges tight.
- Until M is perfect

Procedure

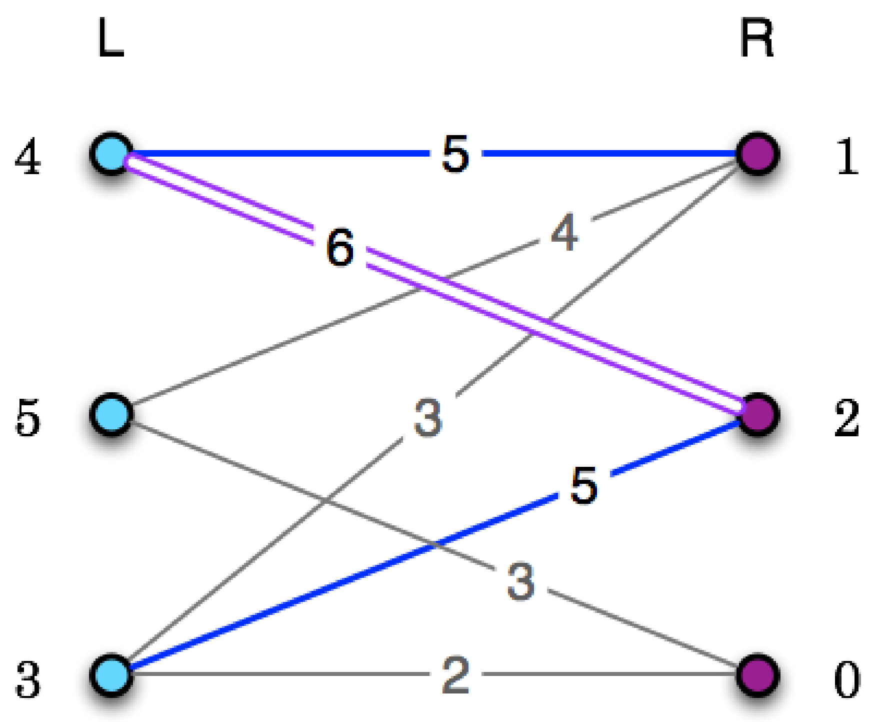
- Let $y(u)=N$, $y(v)=0$ ($u \in L$, $v \in R$)
- Repeat
  - Augment M in G_y (subgraph of tight edges), until there is no augmenting path any more. (Augmentation step)
  - If M is not perfect, adjust the dual variable y to make more edges tight. (Dual adjustment step)
- Until M is perfect

Augmentation step

- Find G_y (subgraph of tight edges)
  - From the tightness condition, all matching edges are in G_y
- Finding augmenting path as in cardinality matching
- Until there is no augmenting paths any more.

Augmentation step

- An example:



Augmentation step

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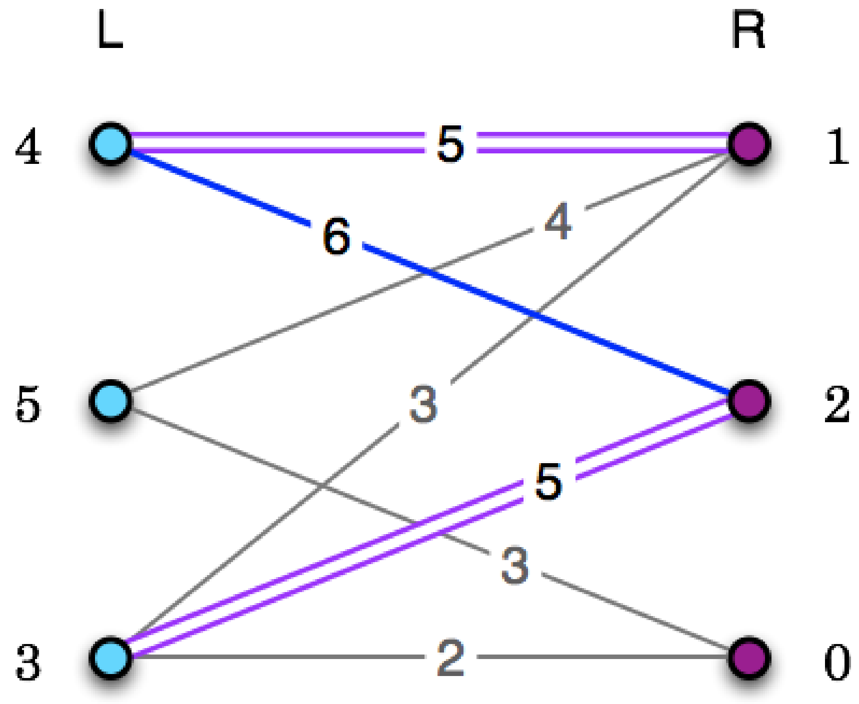


Augmentation Step

- We can use breath-first search to find augmenting paths
- It takes O(m) time for one path.

Dual-adjustment step

- We assign directions to edges in G_y and get G_y':
  - Non-matching edges: from L to R
  - Matching edges from R to L
  - A path between free vertices of L and R in G_y' $\Leftrightarrow$ An augmenting path in G_y

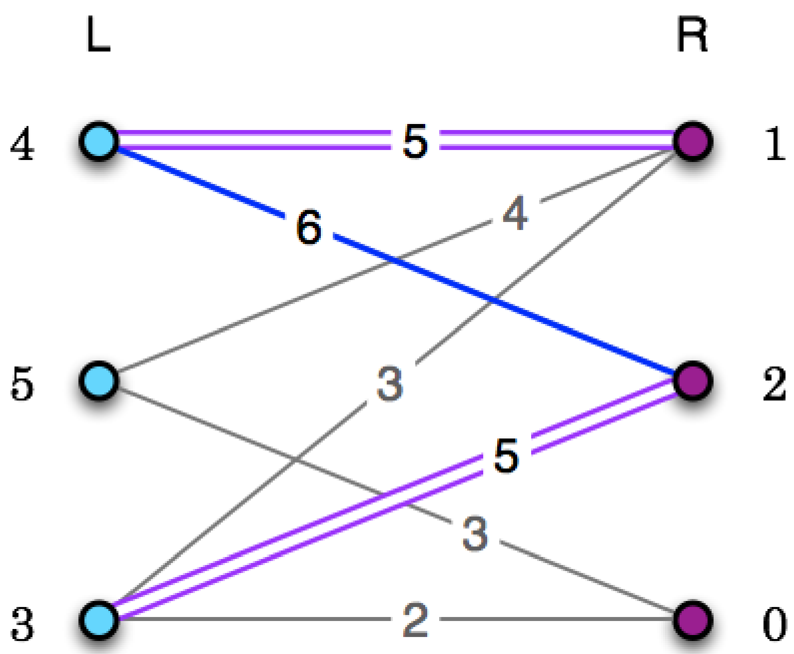
Dual-adjustment step

- We assign directions to edges in G_y and get G_y':
  - Non-matching edges: from L to R
  - Matching edges from R to L
  - A path between free vertices of L and R in G_y' ⇔ An augmenting path in G_y

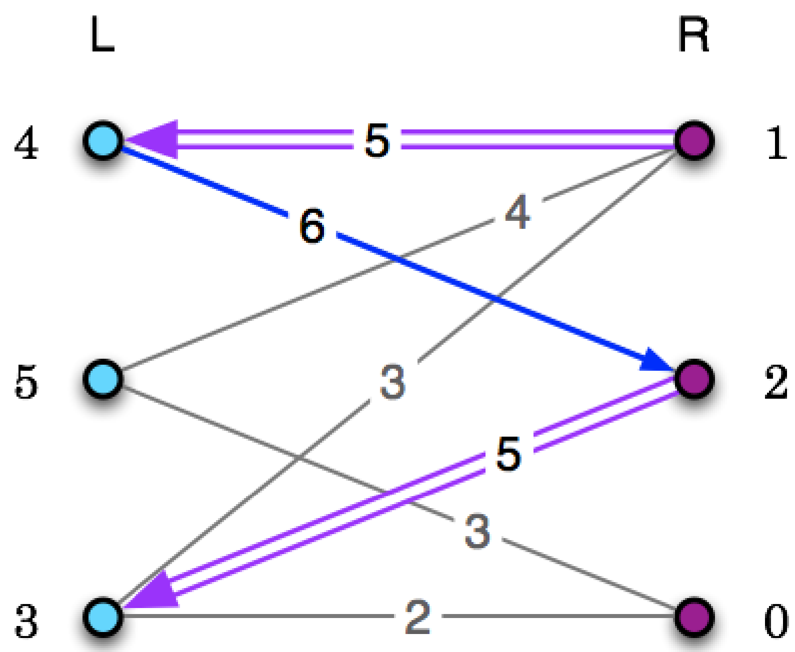
- We have to guarantee there is no augmenting path in G_y before the dual-adjustment
- So there is no directed path between free vertices of L and R in G_y'

An example

G_y



G_y'



Dual-adjustment

- In G_y', find the vertices reachable from free vertices of L, call this set Z
  - Since there is no directed path between free vertices of L and R in G_y', Z does not contain free vertices of R

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- Let $y(u)=y(u)-\Delta$ for $u \in L \cap Z$
- Let $y(v)=y(v)+\Delta$ for $v \in R \cap Z$
  - Δ can bring more tight edges without breaking the domination condition
  - For integer-weighted graph, we can set $\Delta=1$

An example

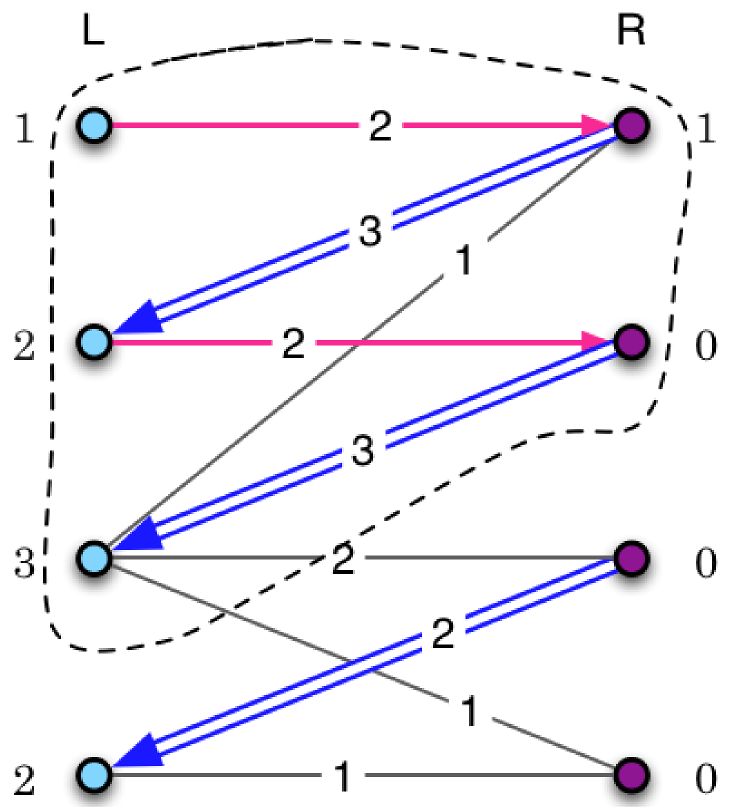
- Tight edges
- Matching edges

(Dual adjustment step)

Let Z be the set of vertices reachable from free vertices of L

Let y(u)=y(u)-Δ for u∈L∩Z

Let y(v)=y(v)+Δ for v∈R∩Z



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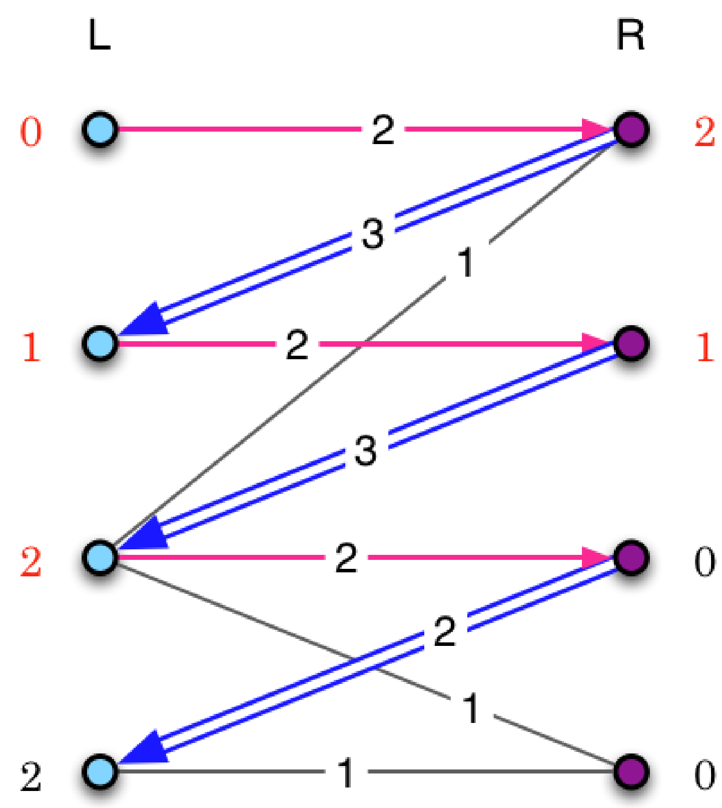
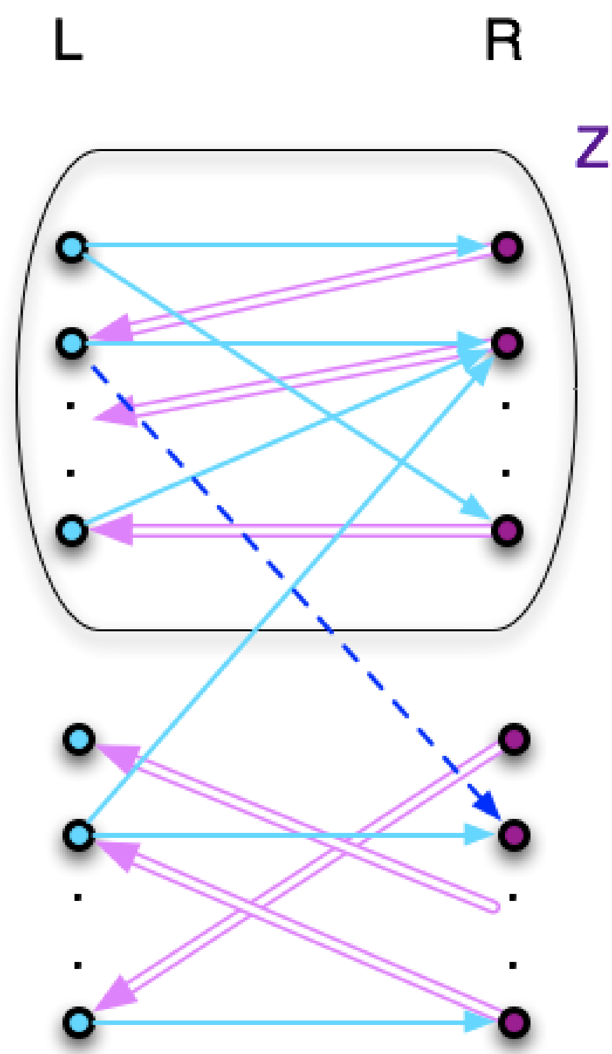


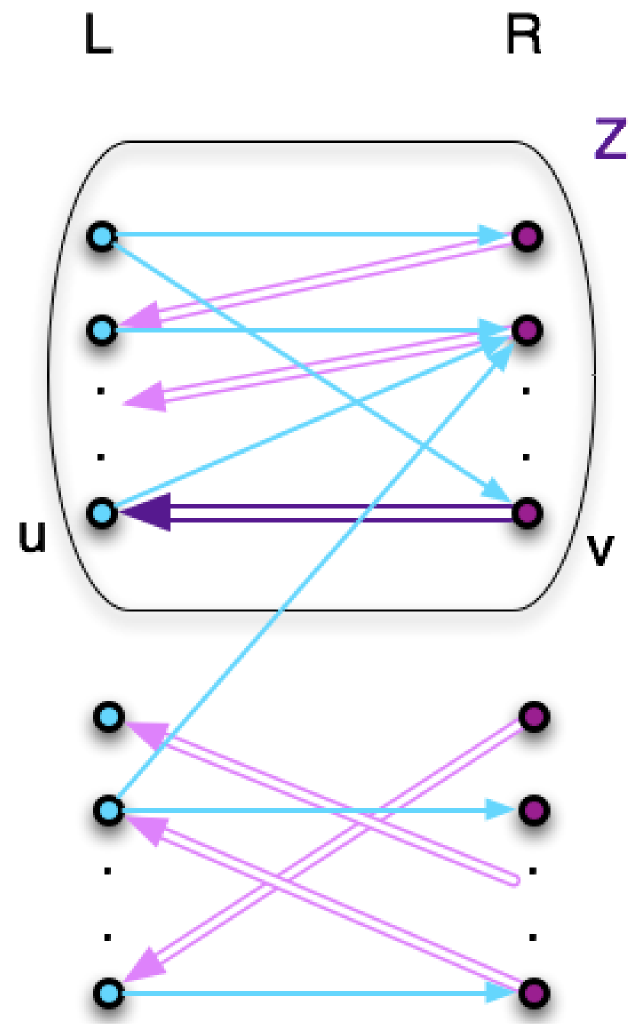
Correctness

- Z is the set of vertices reachable from free vertices of L
- all vertices in L-Z are matched
- all vertices in R∩Z are matched



Correctness

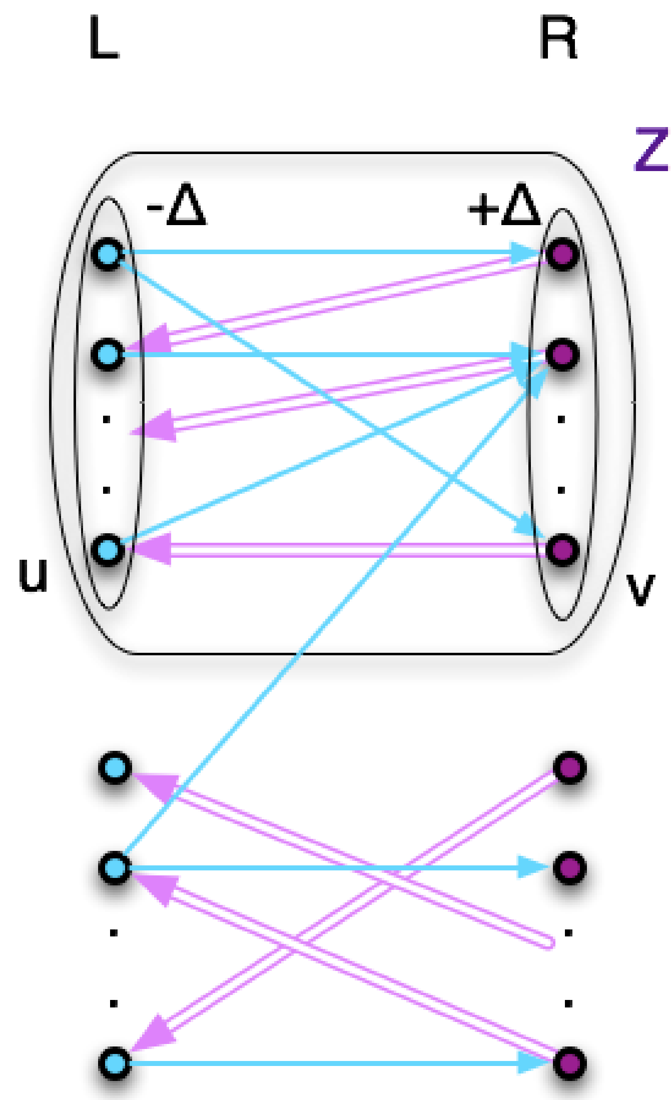
- Z is the set of vertices reachable from free vertices of L
- for a matching edge (u,v), either:
  - u and v are both in Z
  - u and v are neither in Z
  - (If v is in Z, u must be in Z)
  - (u can only be reached from v)



Correctness

- Z is the set of vertices reachable from free vertices of L
- for a matching edge (u,v), either:
  - u and v are both in Z
  - u and v are neither in Z
- So after the dual-adjustment, all matching edges still satisfy

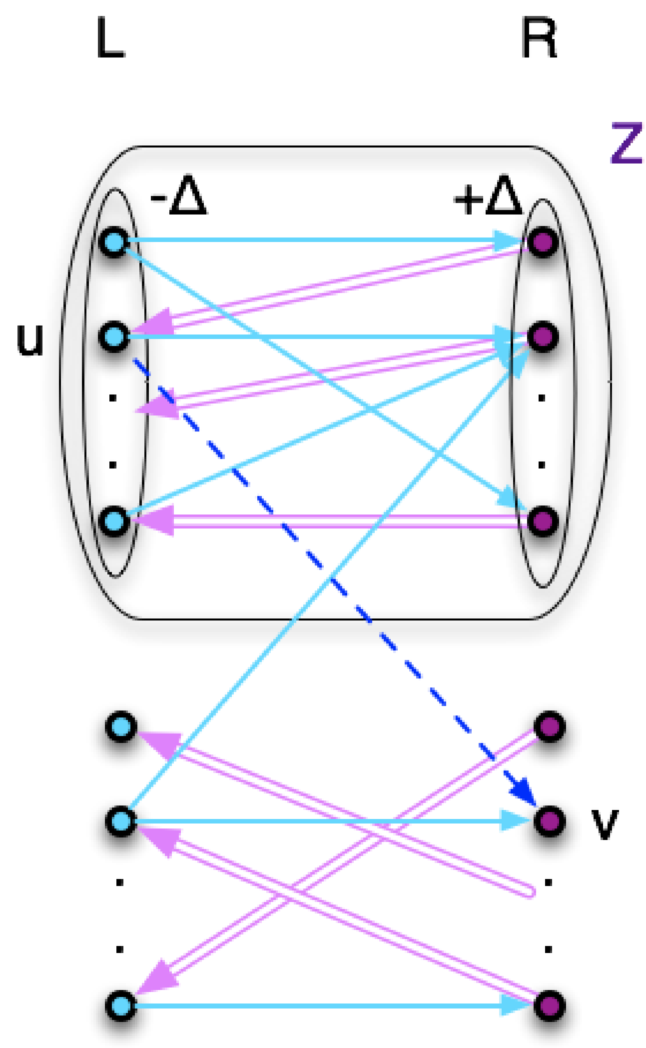
  y(u)+y(v)=w(u,v)



For non-matching edges

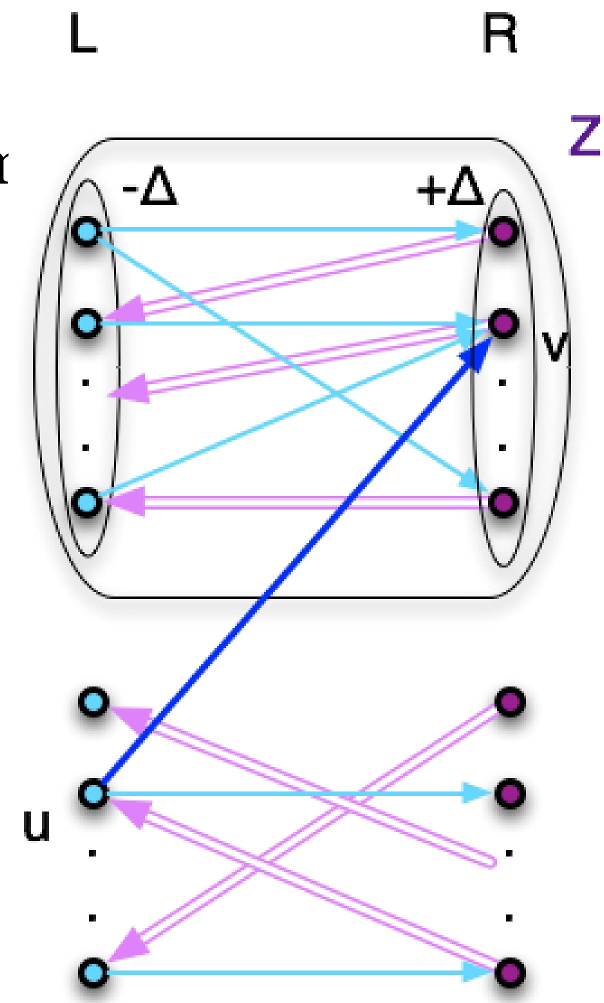
- Z is the set of vertices reachable from free vertices of L by tight edges

- There is no tight edges (u,v) from L∩Z to R-Z
  - Otherwise v will be in Z



For non-matching edges

- Z is the set of vertices reachable from free vertices of L by tight edges
- There is no tight edges (u,v) from L∩Z to R-Z
- For edges (u,v) from L-Z to R∩Z
  - Only v increase
  - The domination condition $y(u)+y(v) \geq w(u,v)$ still holds



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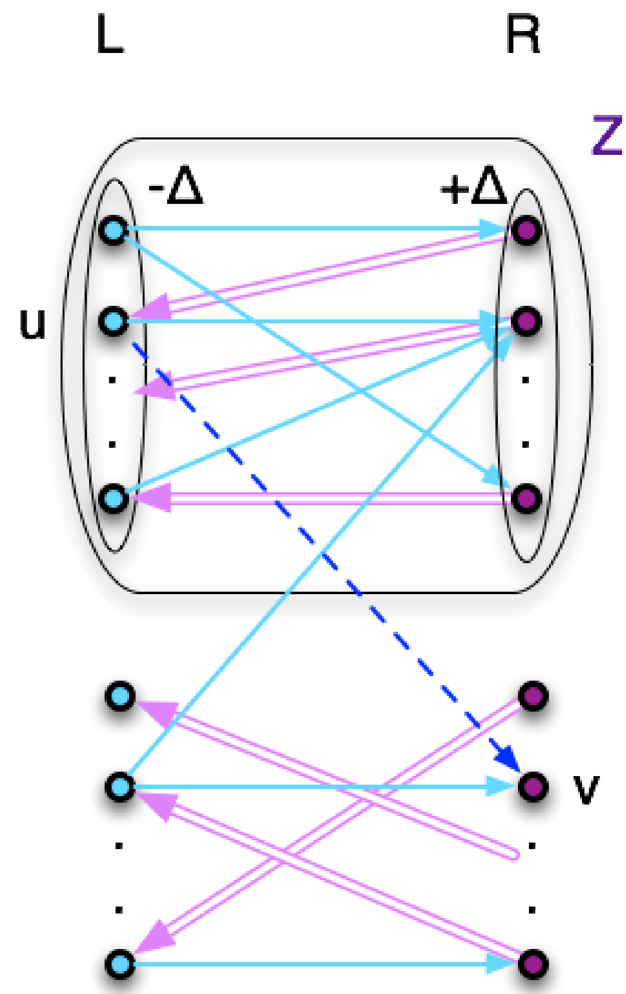
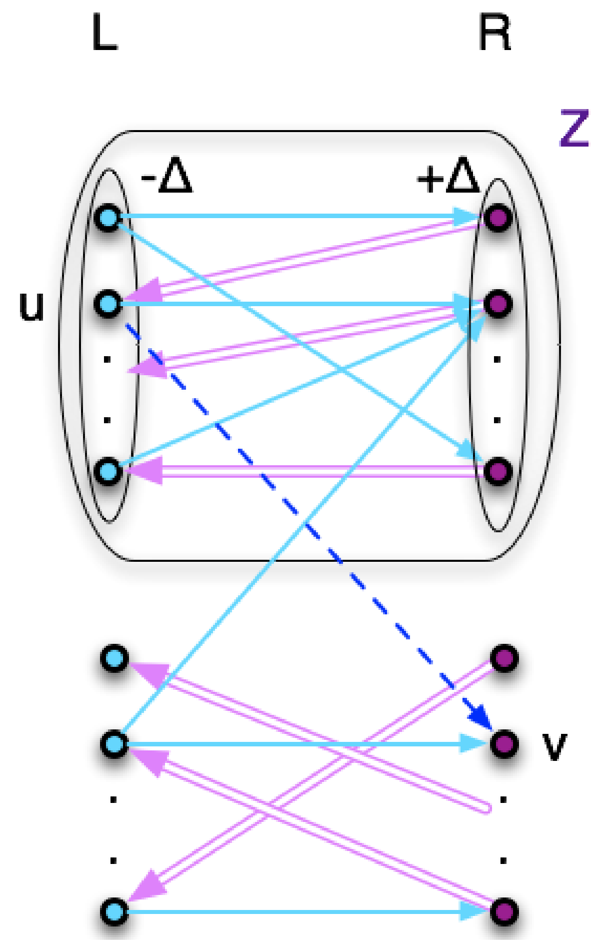
- Z is the set of vertices reachable from free vertices of L by tight edges
- There is no tight edges (u,v) from L∩Z to R-Z

- So the amount of adjustment $\Delta=\min\{y(u)+y(v)-w(u,v) \mid u \in L \cap Z, v \in R-Z\}$
  - So we can have more tight edges, and Z will get larger.



For non-matching edges

- So the amount of adjustment $\Delta = \min\{w(u,v) - y(u) - y(v) \mid u \in L \cap Z,\ v \in R - Z\}$
  - So we can have more tight edges, and Z will get larger.
  - Until some free vertex is added to Z



- Let $y(u)=N$, $y(v)=0$ ($u \in L$, $v \in R$)
- Repeat
  - Augment M in G_y (subgraph of tight edges), until there is no augmenting path any more. (Augmentation step)

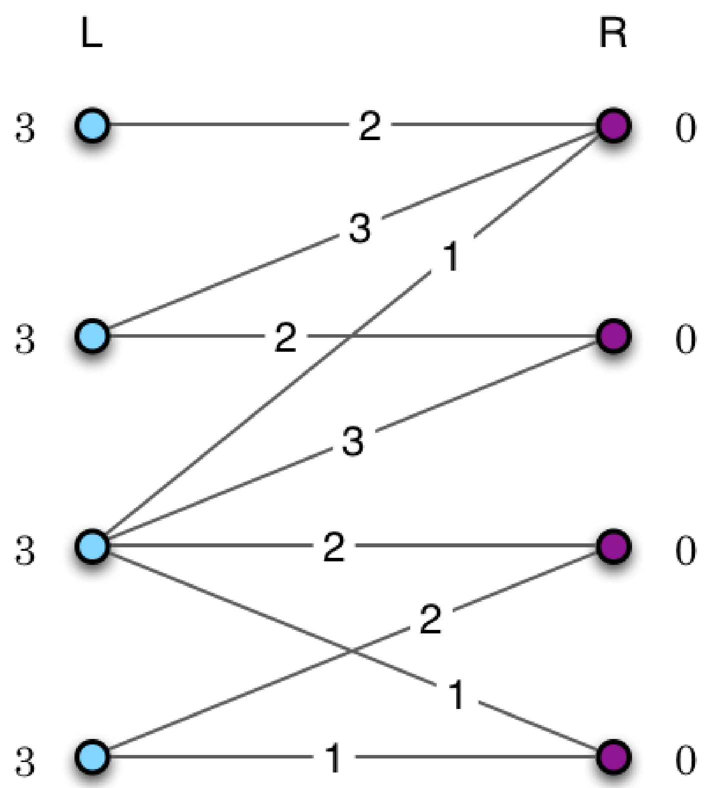
  - If M is not perfect, adjust the dual variable y to make more edges tight. (Dual adjustment step)
    - Let Z be the set of vertices reachable from free vertices of L
    - Let $y(u)=y(u)-\Delta$ for $u \in L \cap Z$
    - Let $y(v)=y(v)+\Delta$ for $v \in R \cap Z$

- Until M is perfect

Running Time

- M can be augmented n times
- There can be at most O(n) dual-adjustment steps before M can be augmented
  - Every time Z becomes larger
- The time needed by searching for an augmenting path or a dual-adjustment step is O(m)
- The total time is $O(mn^2)$

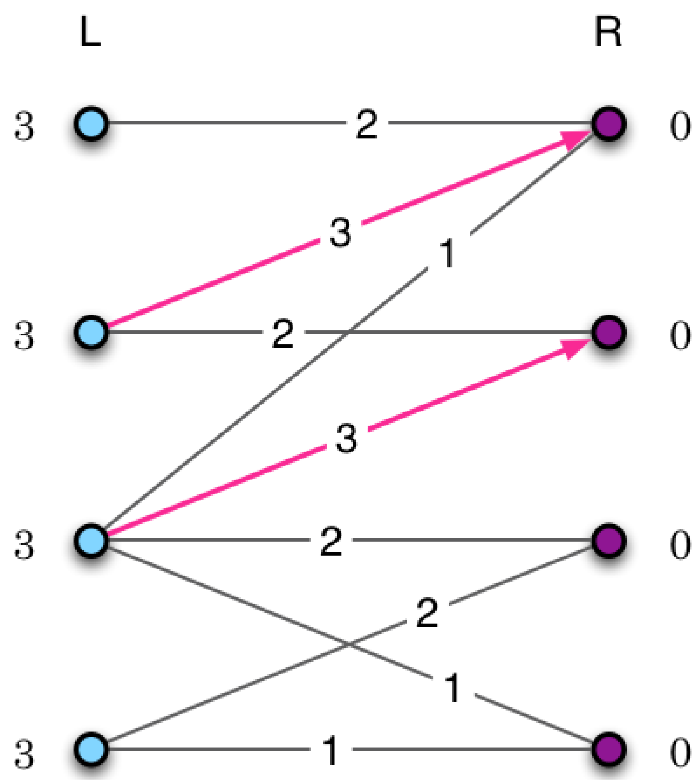
An example



(Augmenting step)
find augmenting path

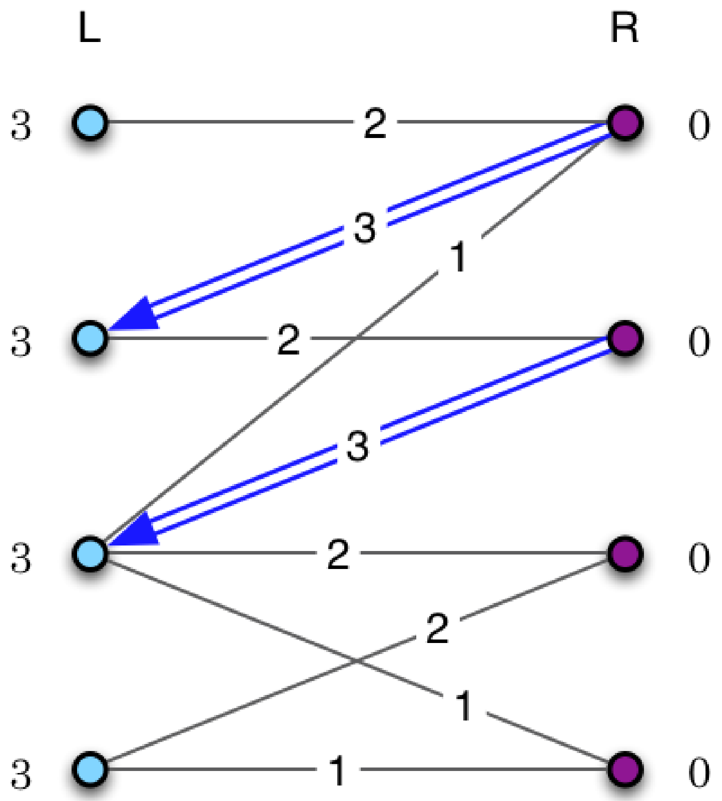
An example

- Tight edges



An example

- Tight edges
- Matching edges



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(Dual adjustment step)

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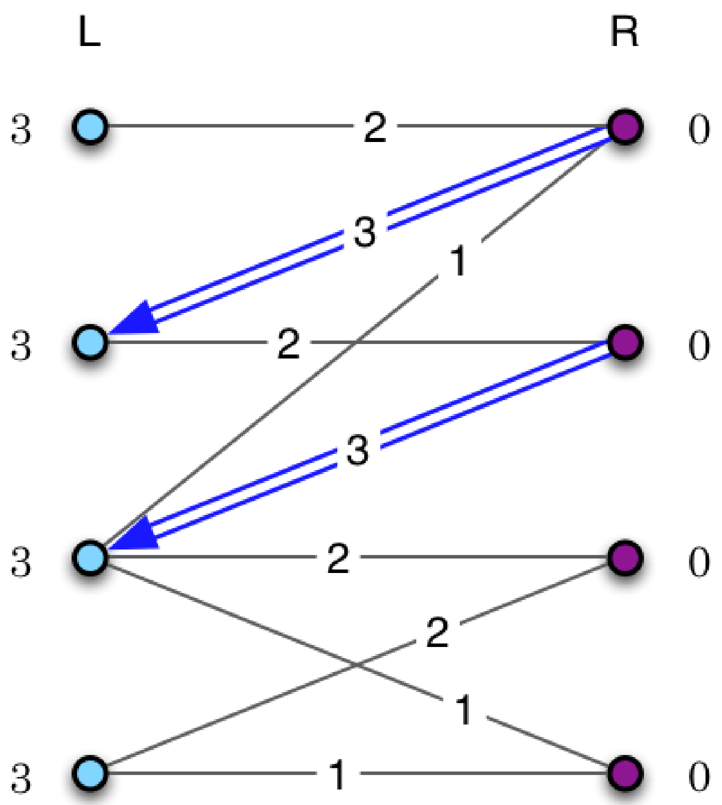

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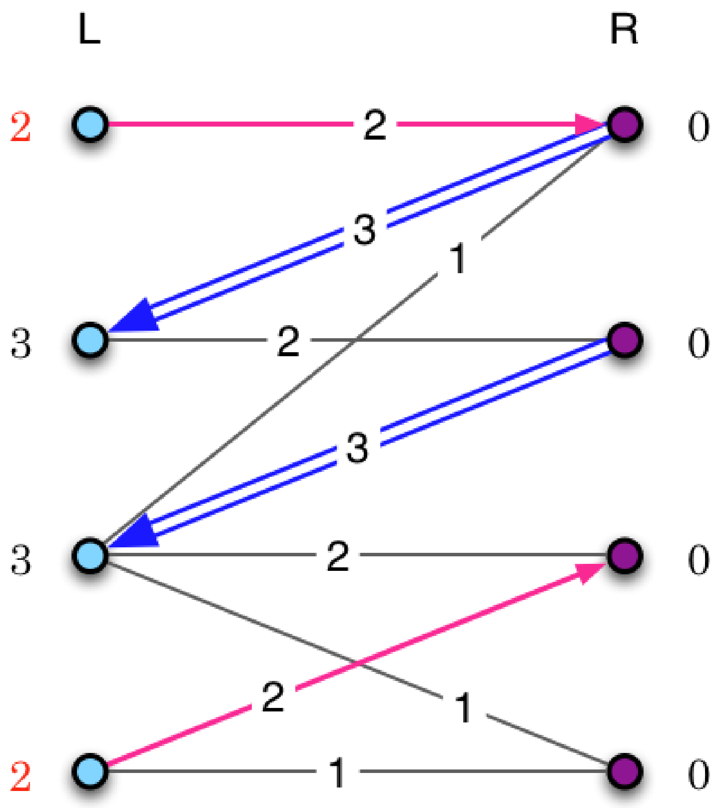

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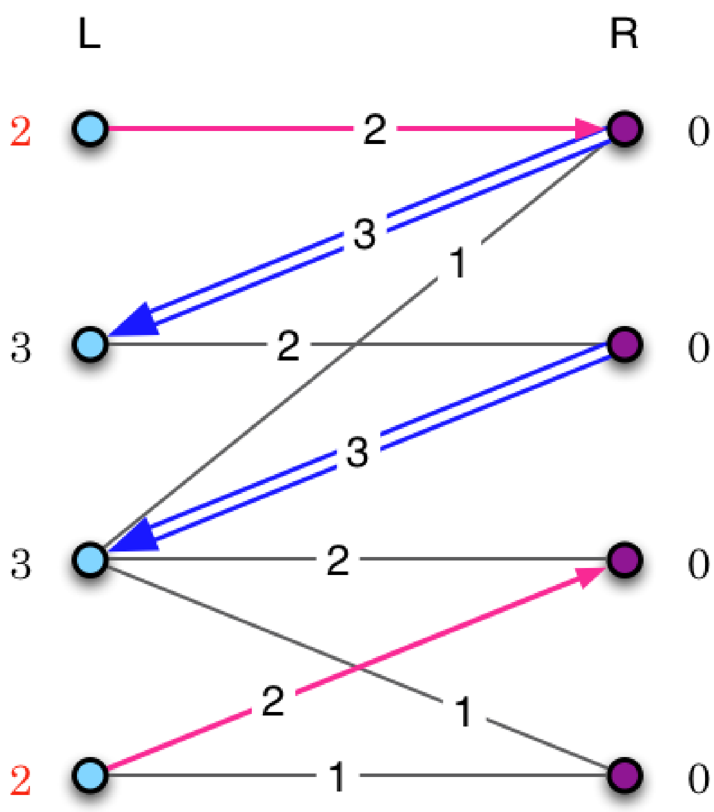

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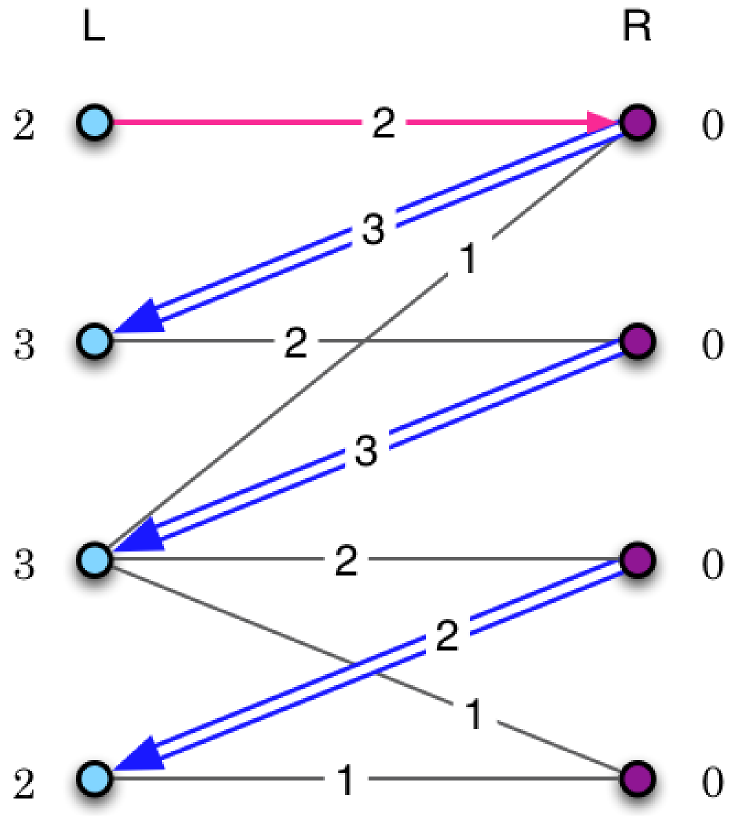

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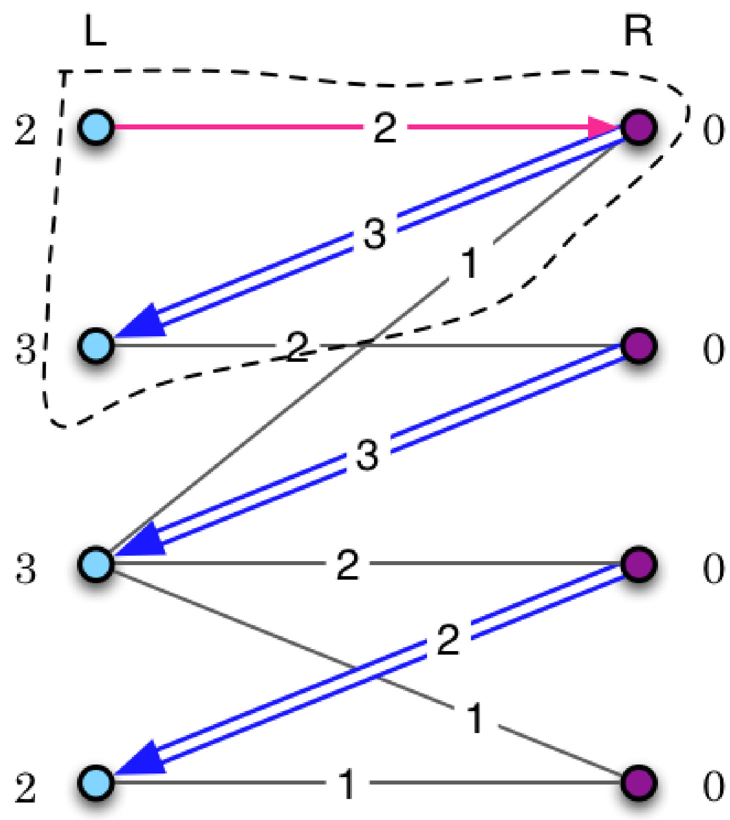

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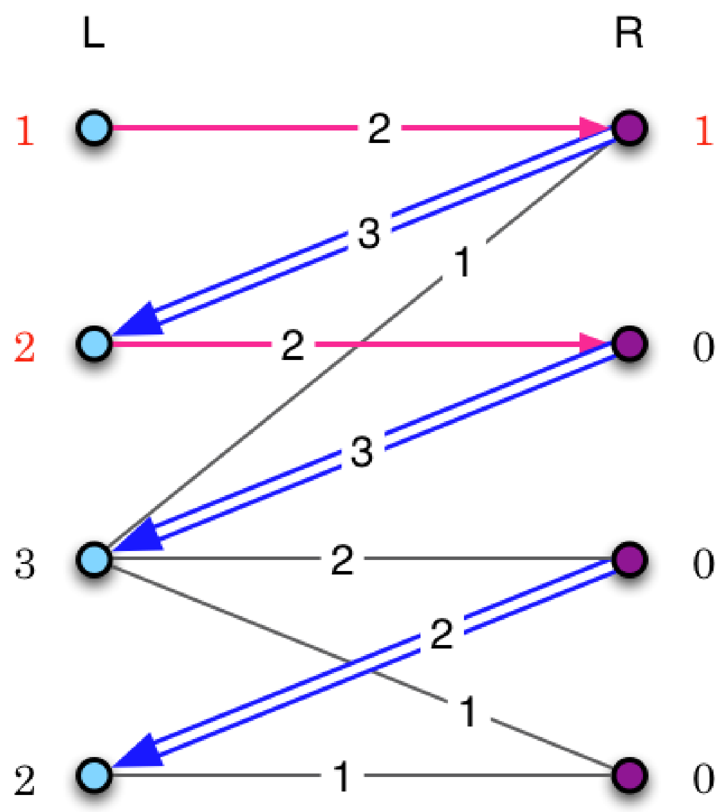

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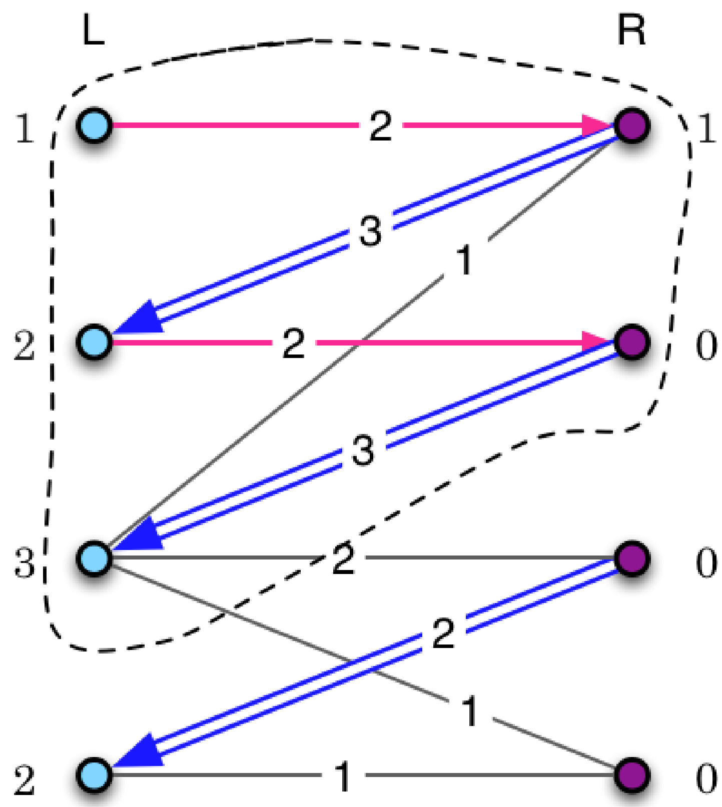
(Dual adjustment step)

Let Z be the set of vertices reachable from free vertices of L

Let $y(u)=y(u)-\Delta$ for $u \in L \cap Z$

Let $y(v)=y(v)+\Delta$ for $v \in R \cap Z$



An example

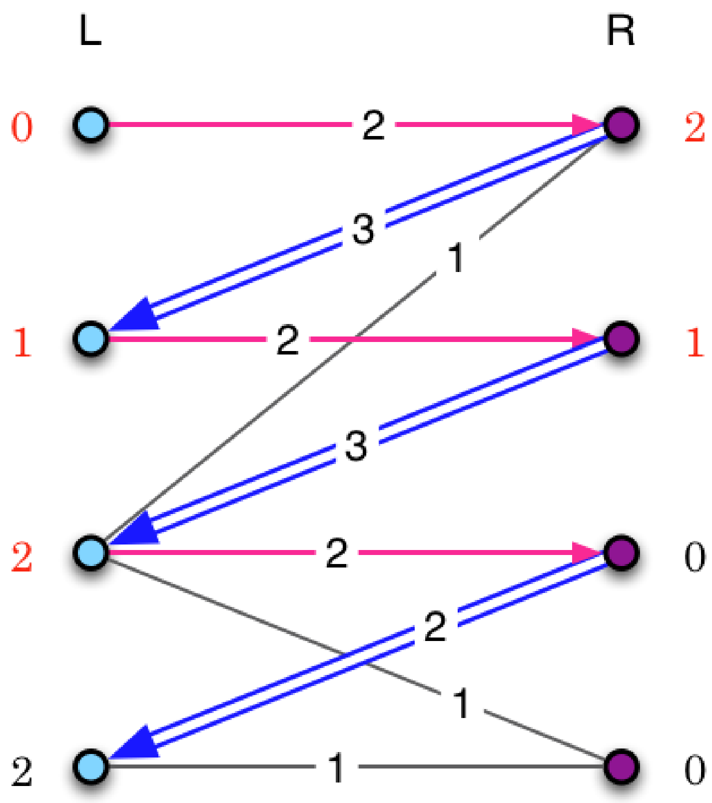
- Tight edges
- Matching edges

(Dual adjustment step)

Let Z be the set of vertices reachable from free vertices of L

Let y(u)=y(u)-Δ   for u∈L∩Z

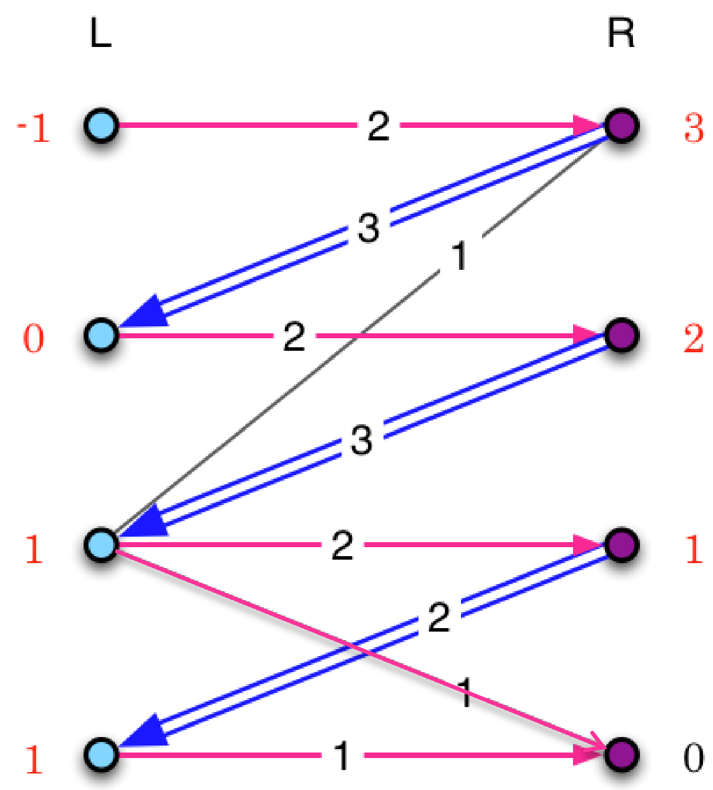
Let y(v)=y(v)+Δ   for v∈R∩Z



(Augmenting step)
find augmenting path

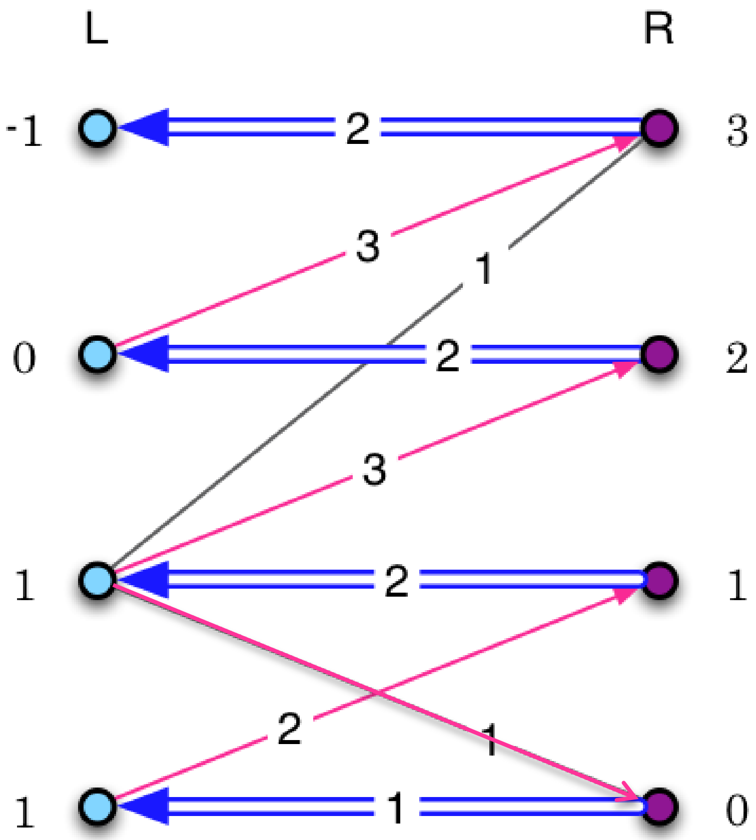
An example

- Tight edges
- Matching edges



Finally

- Note that y-value can be negative



Termination condition

- If we want a maximum (minimum) perfect matching, then we stop when we get a perfect matching M*
- Now $w(M^*) = \sum_{e \in M^*} w(e) = \sum_{v \in L \cap R} y(v)$
- For every other perfect M, $w(M) = \sum_{e \in M} w(e) \leq \sum_{v \in L \cap R} y(v)$
- So $w(M^*) \geq w(M)$

Termination condition

- If we want a maximum matching, then we stop when the free vertices of L have zero y-value.
  - The y-value of free vertices are decreased by the same amount in every step, so they remain equal throughout the algorithm

Termination condition

- Since at the beginning, $y(L)=N$, $y(R)=0$
- In the dual-adjustment step:
  - $y(u)=y(u)-\Delta$ for $u \in L \cap Z$
  - $y(v)=y(v)+\Delta$ for $v \in R \cap Z$
- Z does not contain free vertices in R, otherwise there will be augmenting paths

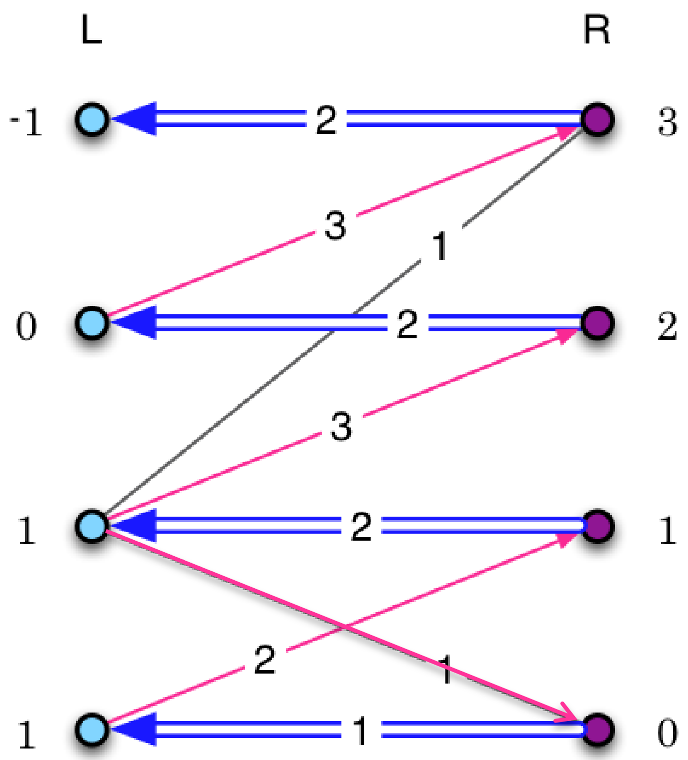
- So the free vertices of R have zero y-value throughout the algorithm

Termination condition

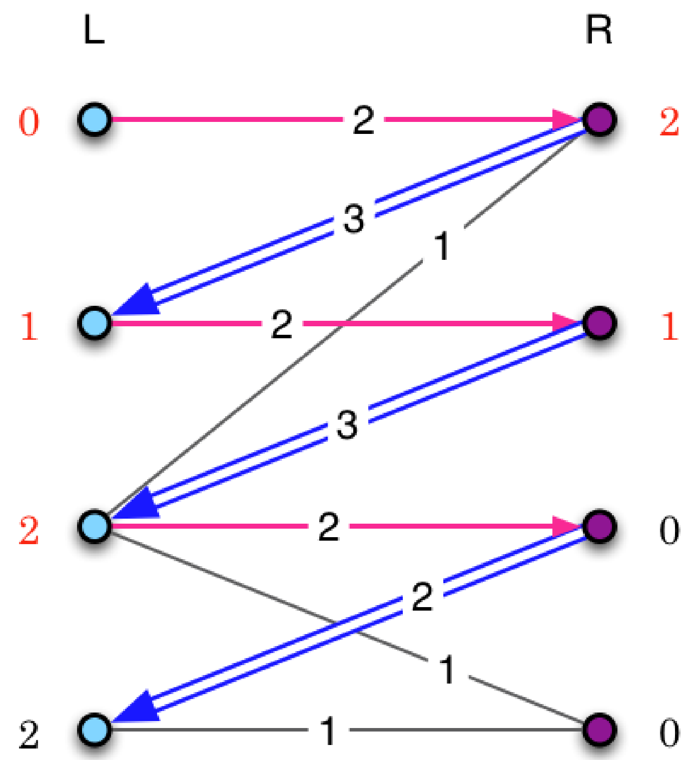
- If we want a maximum matching, then we stop when the free vertices of L have zero y-value, and get M*
- Then all free vertices have zero y-value.
- Now $w(M^*) = \sum_{e \in M^*} w(e) = \sum_{v \in L \cap R} y(v)$
- For every other M, $w(M) = \sum_{e \in M} w(e) \leq \sum_{v \in L \cap R} y(v)$
- So w(M*)≥w(M)

In the example

For maximum perfect matching



For maximum matching



Approximate matching (optional)

- Add a little relaxation on the tightness condition
- Converge more quickly

Original conditions

- Throughout the algorithm:
  - $y(e) \geq w(e)$ (domination)
  - $y(e) = w(e)$ if $e \in M$ (tightness)

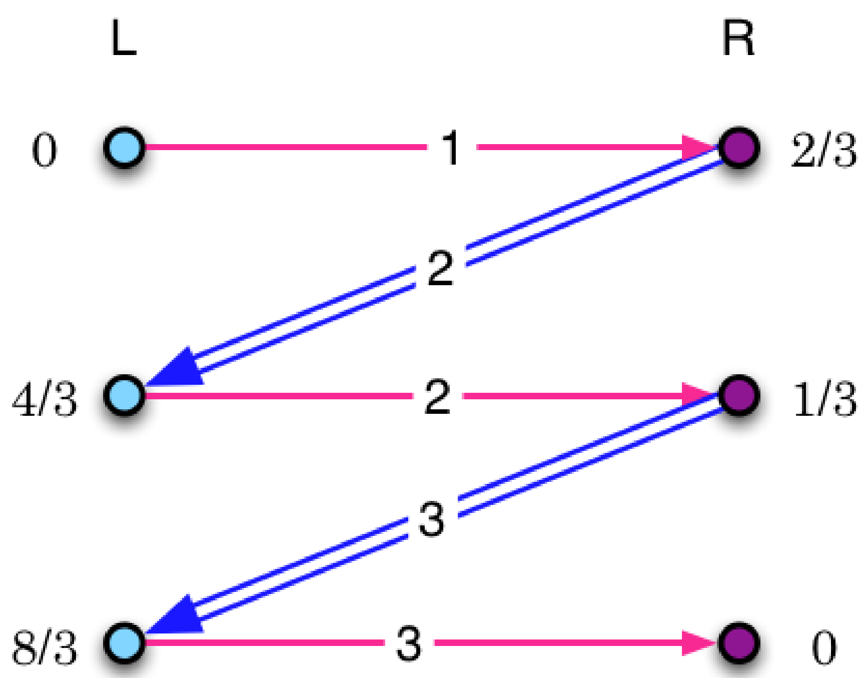
Relaxed conditions

- Throughout the algorithm:
  - $y(e) \geq w(e) - 1/k$ (domination)
  - $y(e) = w(e)$ if $e \in M$ (tightness)

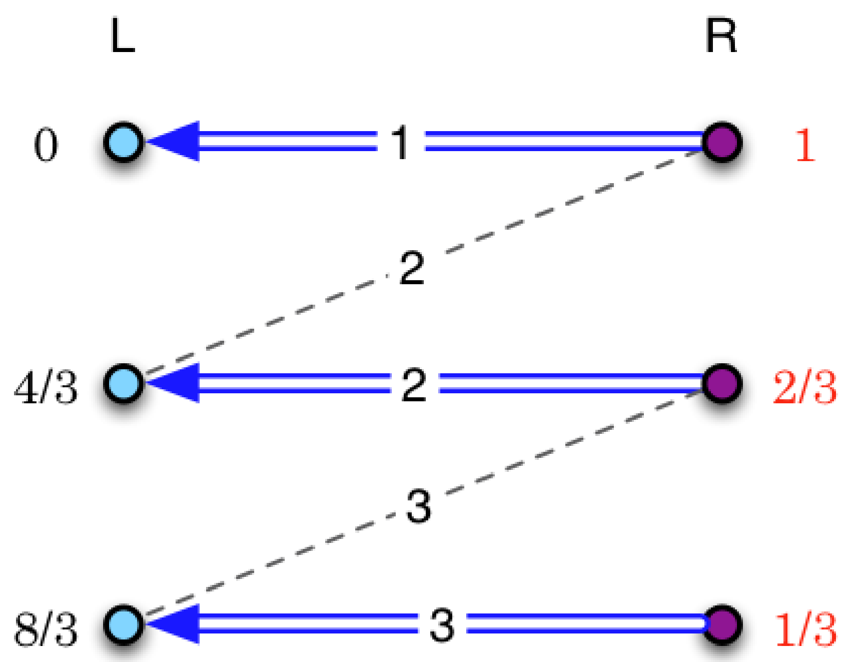
Relaxed conditions

- Throughout the algorithm:
  - $y(e) \geq w(e) - 1/k$ (domination)
  - $y(e) = w(e)$ if $e \in M$ (tightness)
- Then we run the Hungarian search on eligible edges:
  - $y(e) = w(e) - 1/k$ if e not in M
  - all the matching edges

- After augmentation, we add 1/k to the R-side vertex of every new matching edges, so the tightness for matching edges still holds.



- After augmentation, we add 1/k to the R-side vertex of every new matching edges, so the tightness for matching edges still holds.
- So other edges associated with these vertices will not be eligible any more



- After augmentation, we add 1/k to the R-side vertex of every new matching edges, so the tightness for matching edges still holds.
- So other edges associated with these vertices will not be eligible any more
- We just need to find a maximal set of augmenting paths in O(m) time, then there will be no augmenting path before dual-adjustment
- After kN dual-adjustments we can get a (1-1/k)-approximate maximum weighted matching

About the exam time

- All students are now asked to register in HISPOS for the exams for the summer term 2012.
- Please inform the students about the obligatory examination registration.
- In case of problems with the registration, the students can send an email to
  - studium@cs.uni-saarland.de

Next lecture

- Maximum weighted matching in general graphs
- Some applications of matching