Planar Separator Theorem and Its applications

Ran Duan

In this lecture

- Concepts of planar graphs
- Planar separator theorem
- Its applications

Planar Graphs

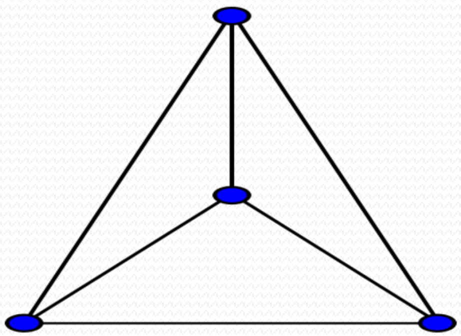
- In graph theory, a **planar graph** is a graph that can be embedded in the plane, i.e. it can be drawn on the plane in such a way that no edges cross each other.

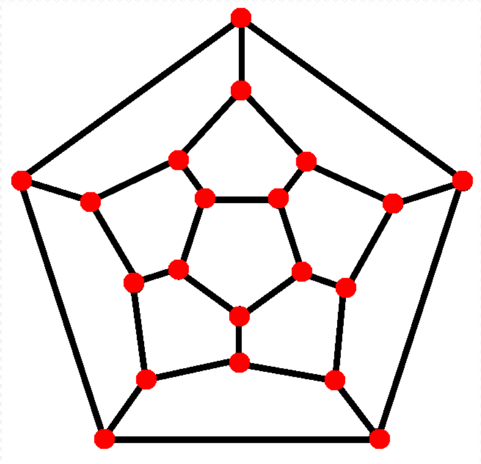
Planar Graphs

- In graph theory, a **planar graph** is a graph that can be embedded in the plane, i.e. it can be drawn on the plane in such a way that no edges cross each other.
- A planar graph already drawn in the plane without edge intersections is called a **plane graph or planar embedding of the graph**

Examples

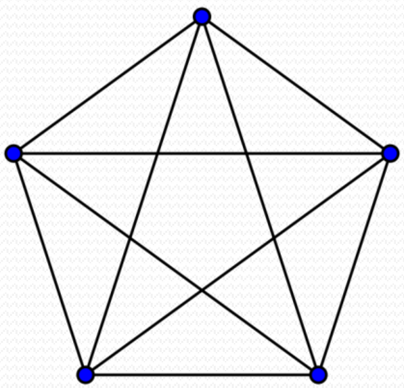
Planar



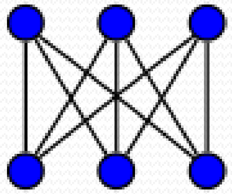


Non-planar

- K_5

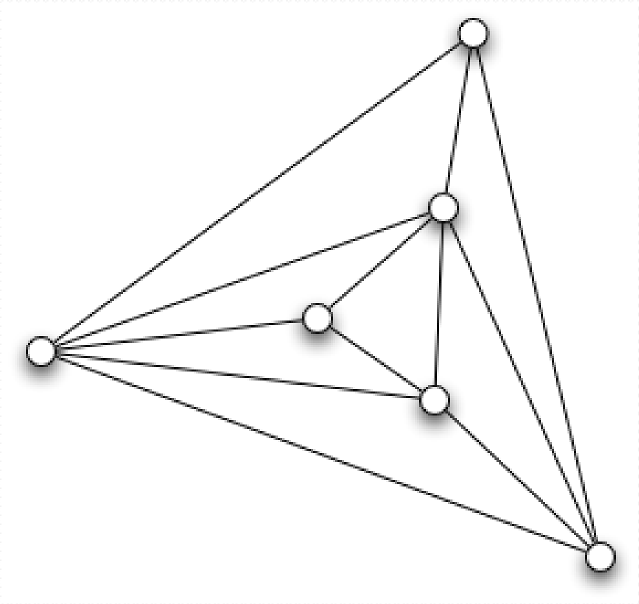


- $K_{3,3}$:



Some facts about planar graph

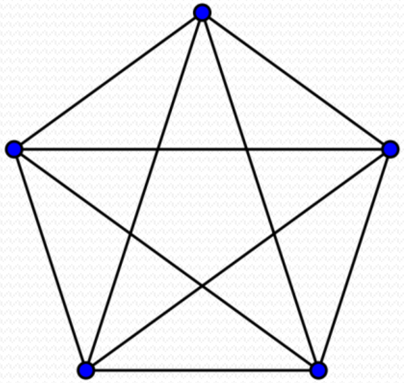
- Any n-vertex planar graph with $n \geq 3$ contains no more than $3n-6$ edges.
- Why?
  - Euler's Formula: $v-e+f=2$
  - Make every face a triangle, then $3f=2e$



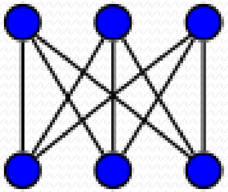
Kuratowski's thereom

- A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

- K_5



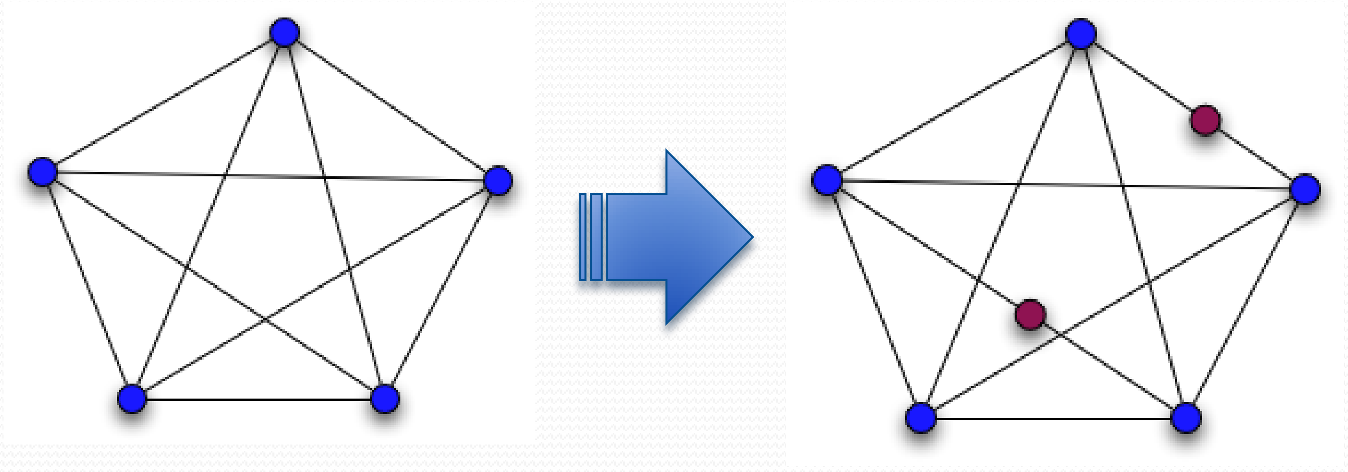
- $K_{3,3}$:



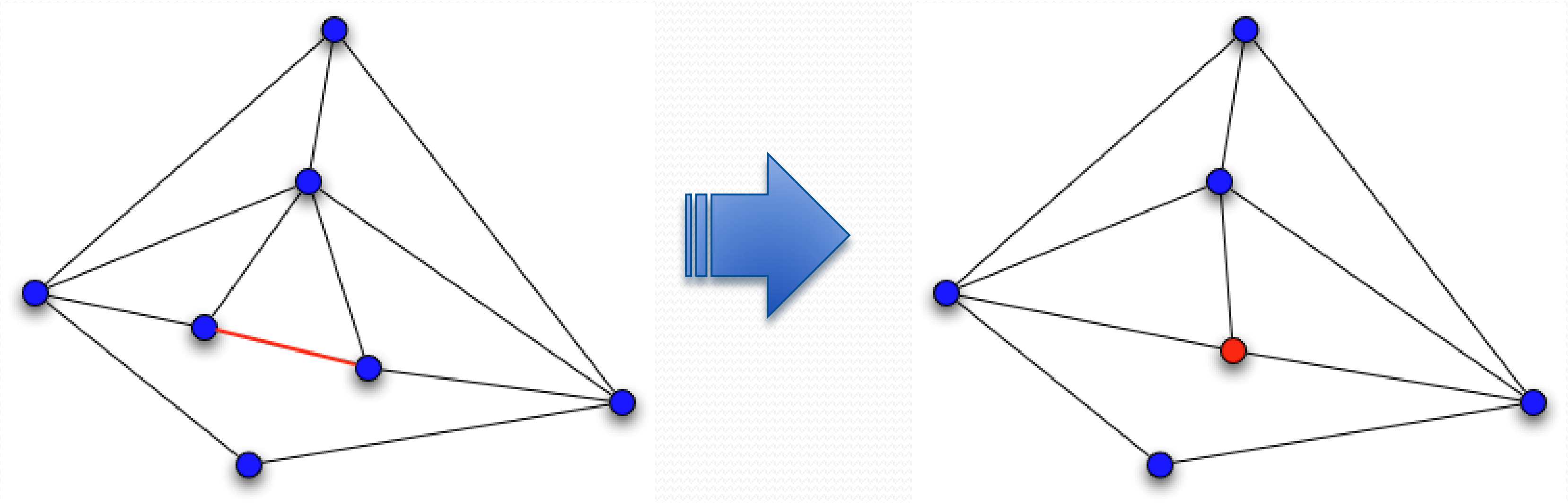
Kuratowski's thereom

- A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

- Subdivision of G
  - insert vertices into edges:



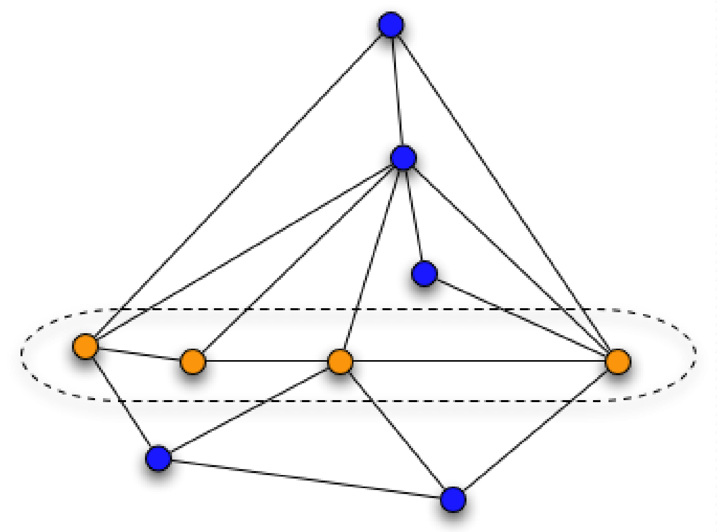
- Let G be any planar graph. Shrinking any edge of G to a single vertex preserves planarity
  - Intuitive result
  - Can be proved by Kuratowski's theorem



- Let G be any planar graph. Shrinking any edge of G to a single vertex preserves planarity
- (Corollary) Let G be any planar graph. Shrinking any connected subgraph of G to a single vertex preserves planarity

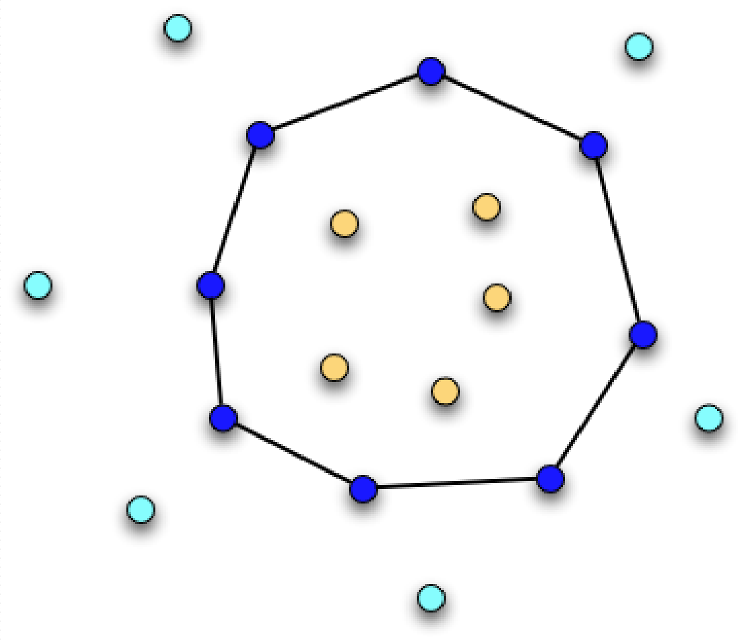
Separator

- The vertices of G are partitioned into three sets: A,B,C, such that no edge joins a vertex in A with a vertex in B, then C is a separator.
  - Useful for "divide-and-conquer" method.
  - usually requires C is small and A,B are at most αn (α is a constant less than 1)



Separator for Planar graph

- In a planar graph, every cycle is a separator:
  - A: vertices inside the cycle
  - B: vertices outside the cycle

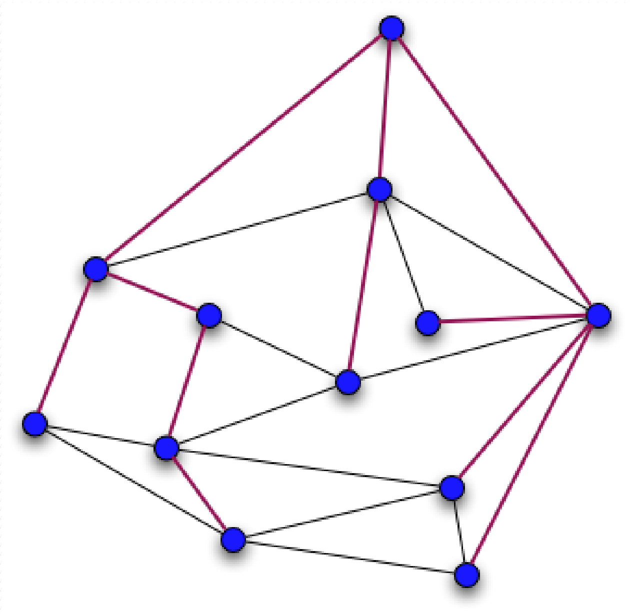


Separator for Planar graph

- $n^{1/2}$-separator theorem: $|A|,|B| \leq \frac{2}{3}n$, $|C| \leq 2\sqrt{2}n^{1/2}$

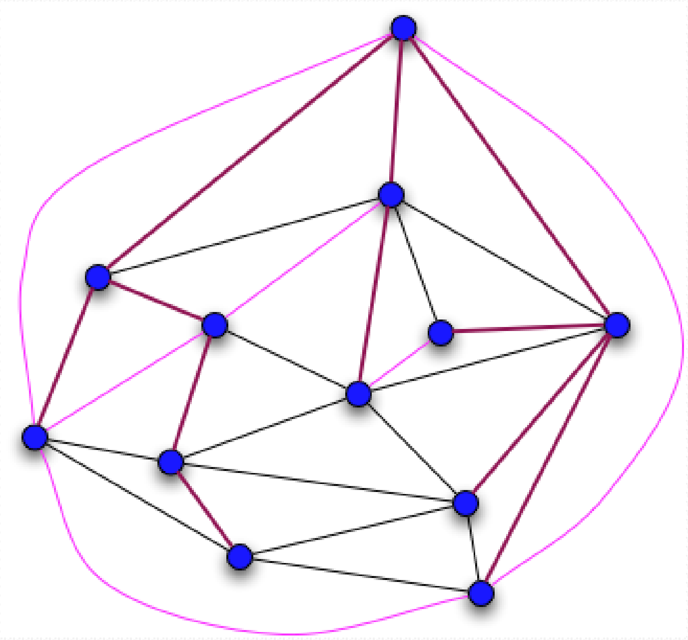
Preliminary theorem:

- Let G be a planar graph with nonnegative vertex costs whose sum ≤1
- If G has a spanning tree T of radius r, then G has a separator C, s.t. neither A nor B has total cost more than 2/3, and C contains at most 2r+1 vertices.



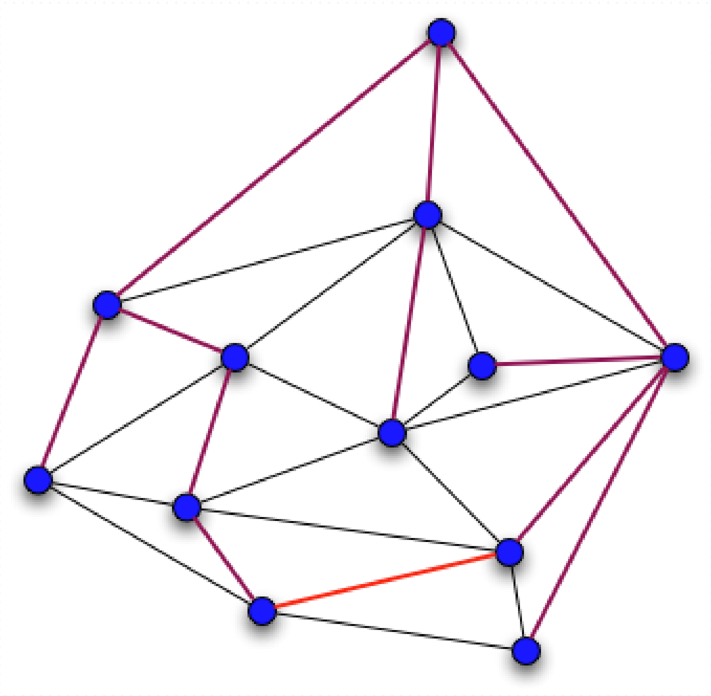
Proof

- Assume no vertex has cost more than 1/3
- First, make each face triangle by adding additional edges



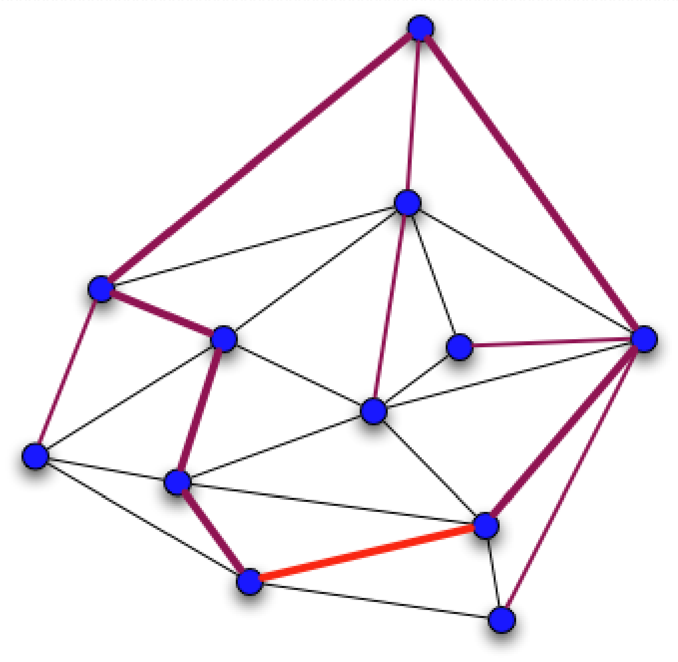
Proof

- First, make each face triangle by adding additional edges
- Any non-tree edge forms a simple cycle with tree edges



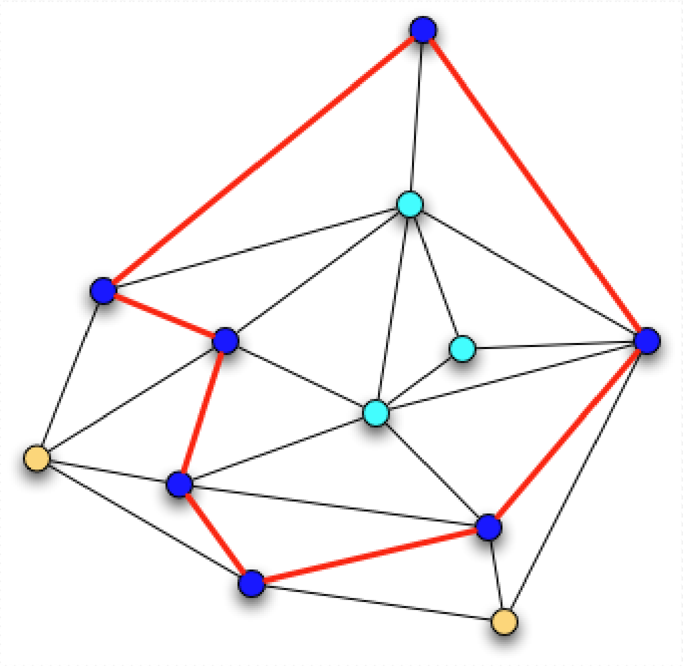
Proof

- First, make each face triangle by adding additional edges
- Any non-tree edge forms a simple cycle with tree edges



Proof

- First, make each face triangle by adding additional edges
- Any non-tree edge forms a simple cycle with tree edges
  - forms a cycle of length at most 2r+1
  - divides the plane into two parts: inside and outside



Proof

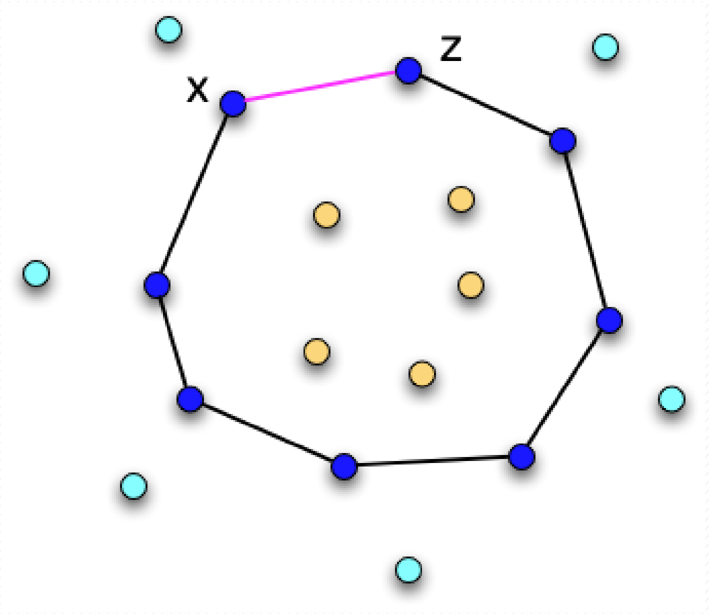
- First, make each face triangle by adding additional edges
- Any non-tree edge forms a simple cycle with tree edges
  - forms a cycle of length at most 2r+1
  - divides the plane into two parts: inside and outside
  - we will show that there exists such a cycle separating the plane so that neither the inside nor the outside contains vertices with total cost more than 2/3.

Proof

- Just find the non-tree edge (x,z) such that its cycle separates the vertices most equally:
  - minimize the max{costs inside cycle, costs outside cycle}
  - break ties by choose the cycle with smallest number of faces on the "max" side
  - if ties remain, choose arbitrarily
- So the cycle with (x,z) is what we want.

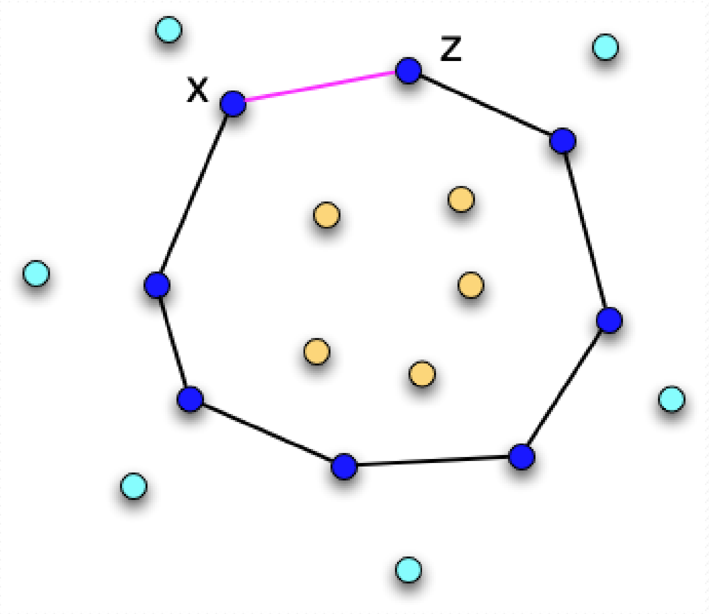
If G has a spanning tree T of radius r, then G has a separator C, s.t. neither A nor B has total cost more than 2/3, and C contains at most 2r+1 vertices.

- Assume the “max” side is the inside
- If the total cost inside is ≤2/3, the claim is true.



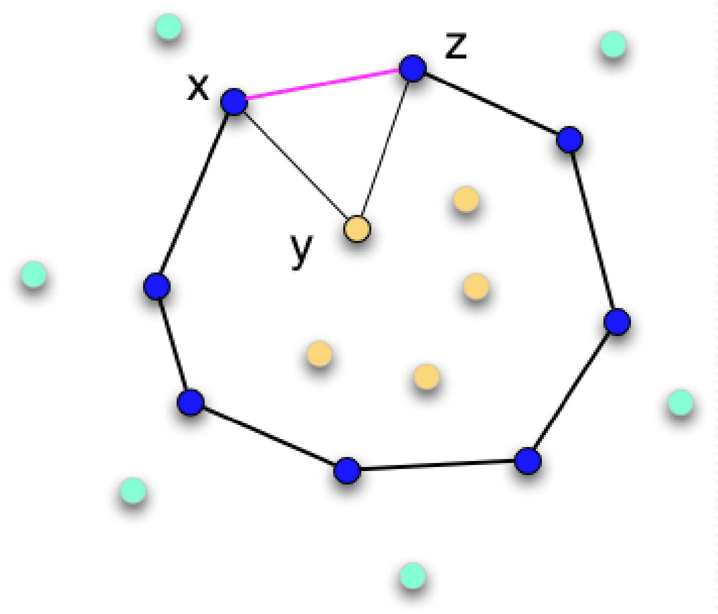
If G has a spanning tree T of radius r, then G has a separator C, s.t. neither A nor B has total cost more than 2/3, and C contains at most 2r+1 vertices.

- Consider the total cost of vertices inside the cycle is >2/3
- We will show that it contradicts the way we choose (x,z)



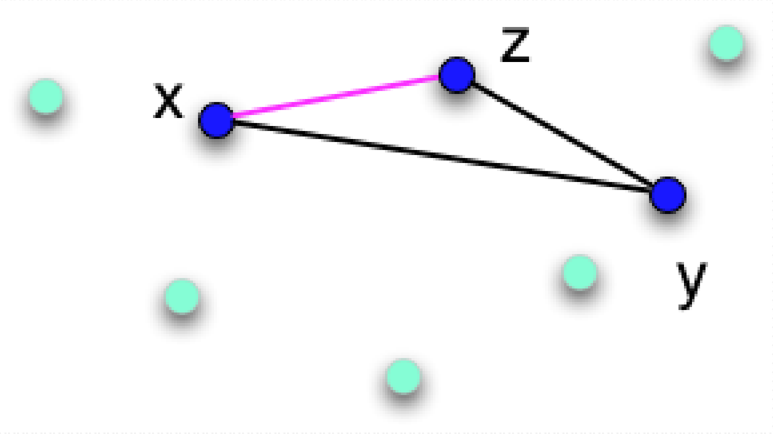
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- consider the triangular face which has (x,z) as a boundary edge and lies inside the cycle, let the third vertex by y
- We study it case by case.



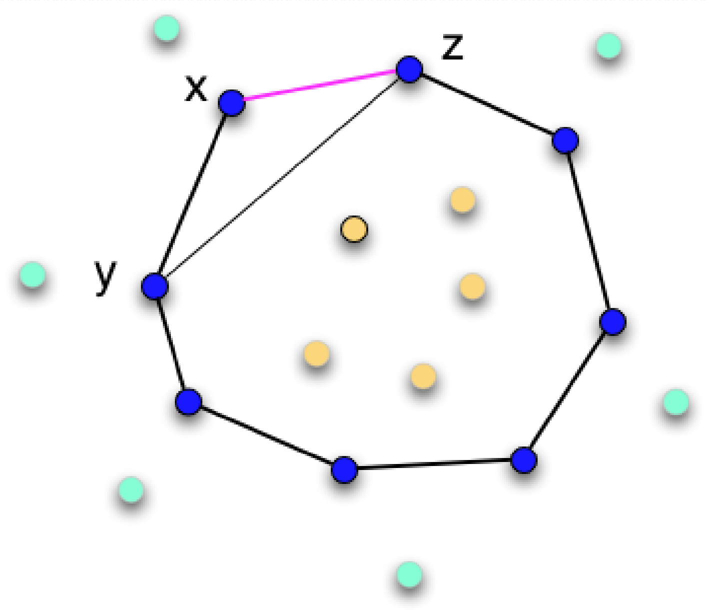
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- 1. Both (x,y) and (y,z) lies on the cycle, then the face (x,y,z) is the cycle, contradicting the inside is the "max" side.
  - since (x,y,z) is one face.



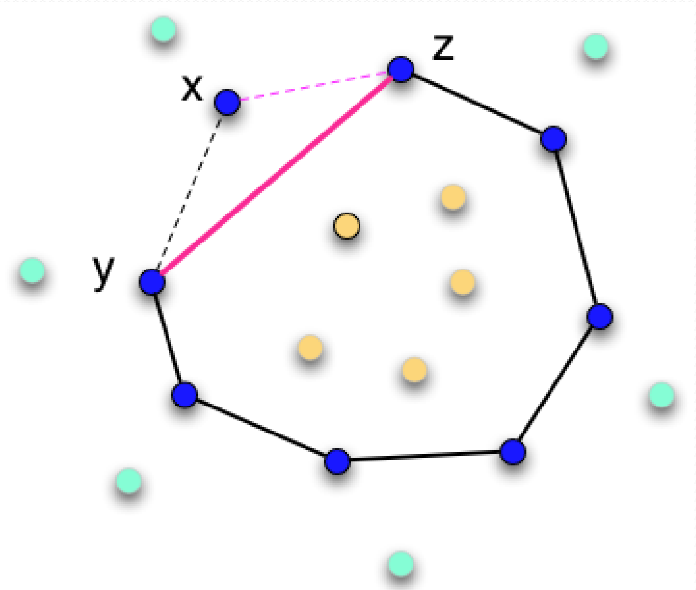
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- 2. One of (x,y) and (y,z) lies on the cycle, assume it is (x,y)



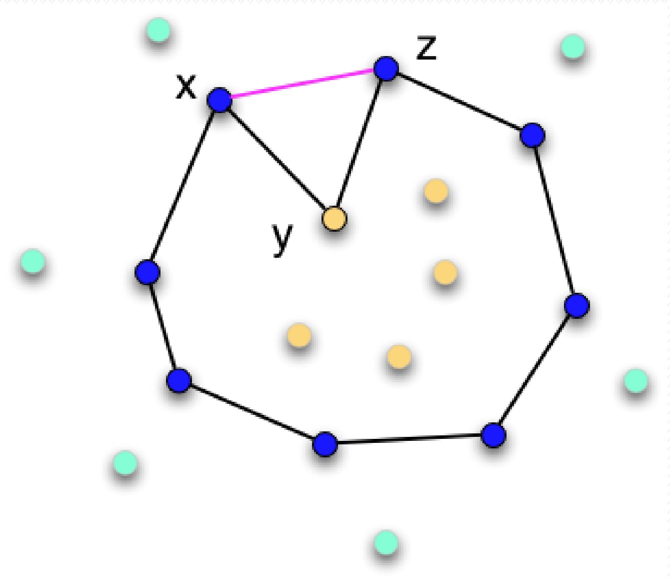
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- 2. One of (x,y) and (y,z) lies on the cycle, assume it is (x,y)
  - Then (y,z) is a non-tree edge defining a cycle with the same vertices on the "max" side but with one less face.
  - contradicting



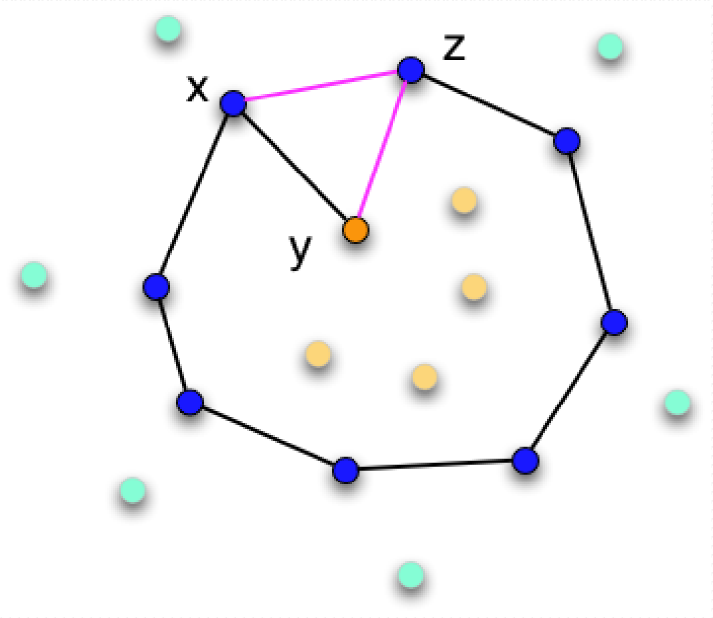
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- 3. Neither (x,y) nor (y,z) lies on the cycle
  - It is impossible that both (x,y) and (y,z) are tree edges, since the tree contains no cycle



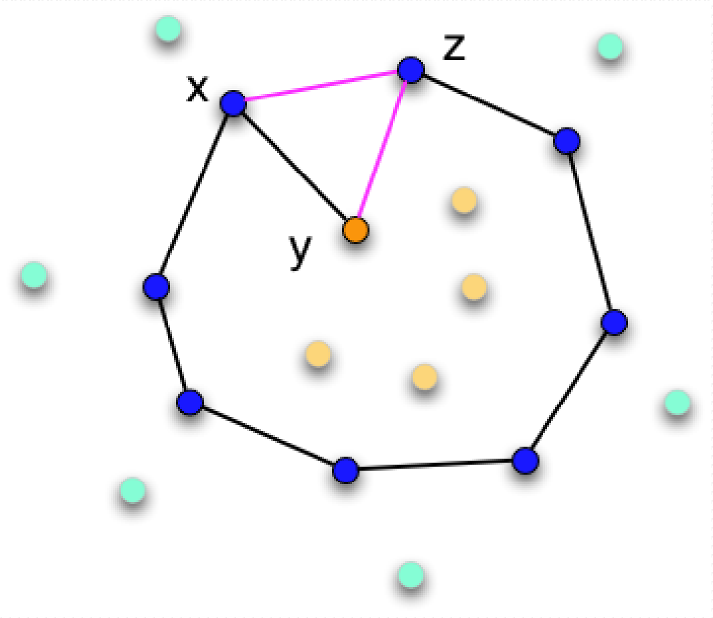
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- 3. Neither (x,y) nor (y,z) lies on the cycle
  - One of them is a tree edge, assume it is (x,y)
  - The cycle with (y,z) has one less vertex y and one less face inside than the cycle with (x,z)



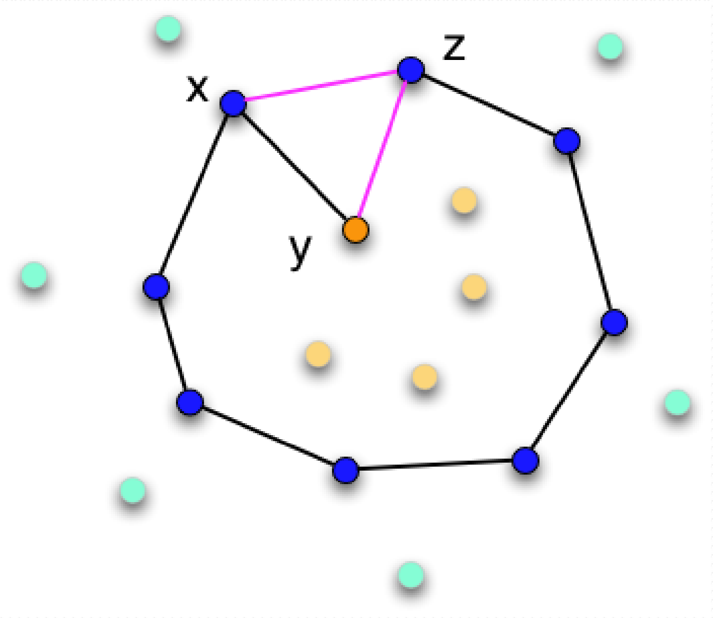
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- The cycle with (y,z) has one less vertex y and one less face inside than the cycle with (x.z)
- If the cost inside the (y,z) is greater than the cost outside, (y,z) would have been chosen in place of (x,z)



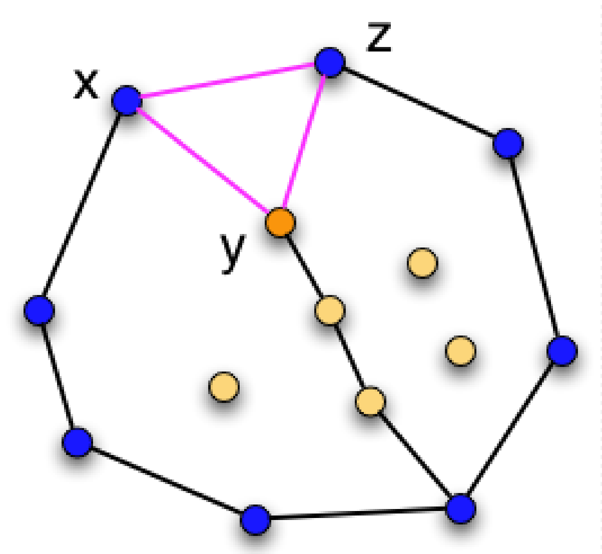
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- Otherwise: since the cost inside the (x,z) cycle is ≥2/3, and the cost of y is ≤1/3, so the cost inside the (y,z) cycle is ≥1/3.
- So (y,z) cycle would have been chosen instead of (x,z)



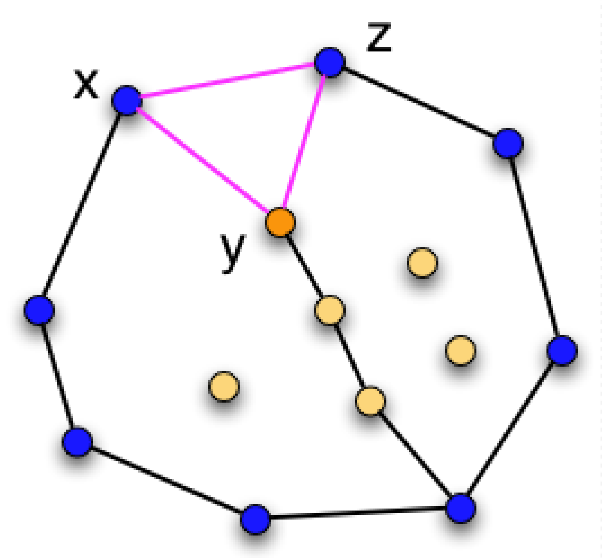
- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- 3. Neither (x,y) nor (y,z) lies on the cycle
  - Neither of them is a tree edge, then each of (x,y) and (y,z) defines a cycle.
  - every vertex inside the (x,z) cycle would: inside the (x,y) cycle, inside the (y,z) cycle or on the boundary



- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- Neither of them is a tree edge, then each of (x,y) and (y,z) defines a cycle.
- every vertex inside the (x,z) cycle would: inside the (x,y) cycle, inside the (y,z) cycle or on the boundary
- Choose the cycle (say (x,y)-cycle) with more total cost inside, since the cost inside the (x,z) cycle >2/3, the total cost inside the (x,y)-cycle and itself >1/3, so the cost outside (x,y) cycle <2/3.

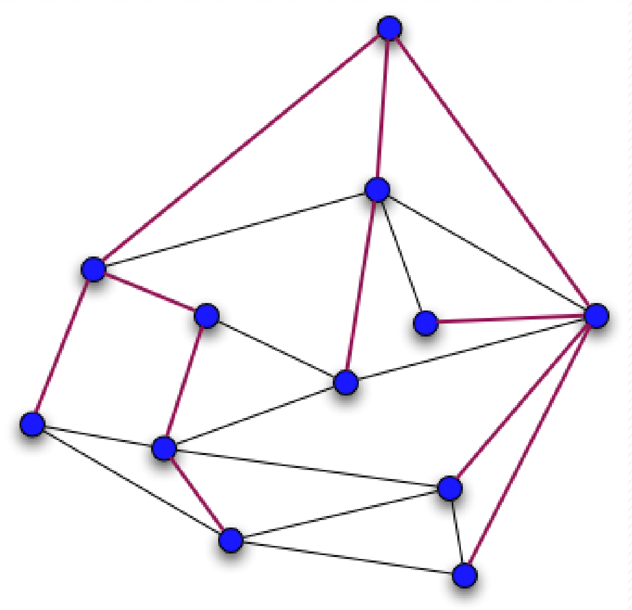


- minimize the max{costs inside cycle, costs outside cycle}
- break ties by choosing the cycle with smallest number of faces on the "max" side

- Neither of them is a tree edge, then each of (x,y) and (y,z) defines a cycle.
- every vertex inside the (x,z) cycle would: inside the (x,y) cycle, inside the (y,z) cycle or on the boundary
- Choose the cycle (say (x,y)-cycle) with more total cost inside, since the cost inside the (x,z) cycle >2/3, the total cost inside the (x,y)-cycle and itself >1/3, so the cost outside (x,y) cycle <2/3.
- If the cost inside (x,y) cycle is greater than outside, (x,y) would have been chosen since the cost inside (x,y) cycle is smaller than the cost inside (x,z) cycle.
- Otherwise the cost inside the (x,y) cycle <1/2, so the (x,y) cycle is what we want.

We have proved:

- Let G be a planar graph with nonnegative vertex costs whose sum ≤1
- If G has a spanning tree T of radius r, then G has a separator C, s.t. neither A nor B has total cost more than 2/3, and C contains at most 2r+1 vertices.



Main theorem:

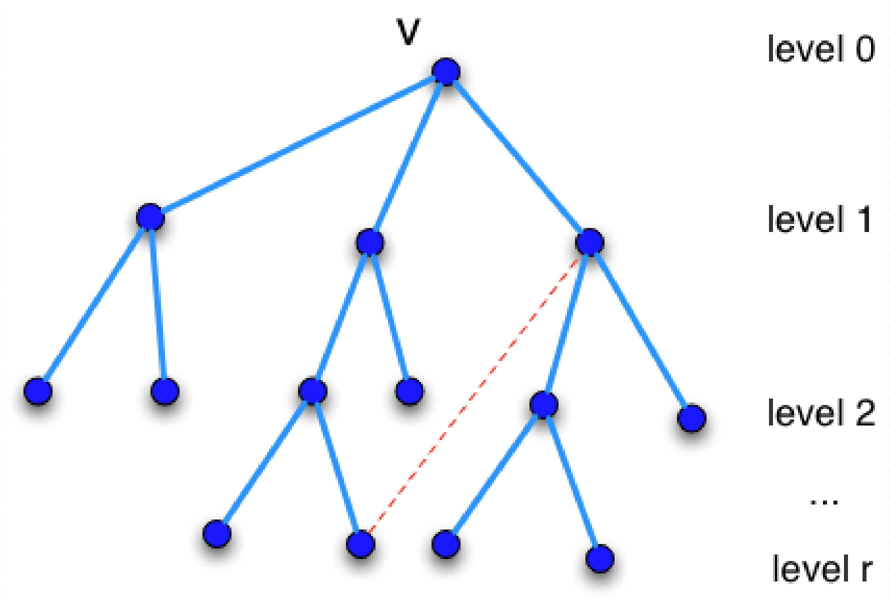
- Let G be a planar graph with nonnegative vertex costs whose sum ≤ 1
- Then the vertices of G can be partitioned into A,B,C
  - no edge joins A and B
  - neither A nor B has total cost $>2/3$
  - $|C| \leq 2\sqrt{2}n^{1/2}$

Main theorem:

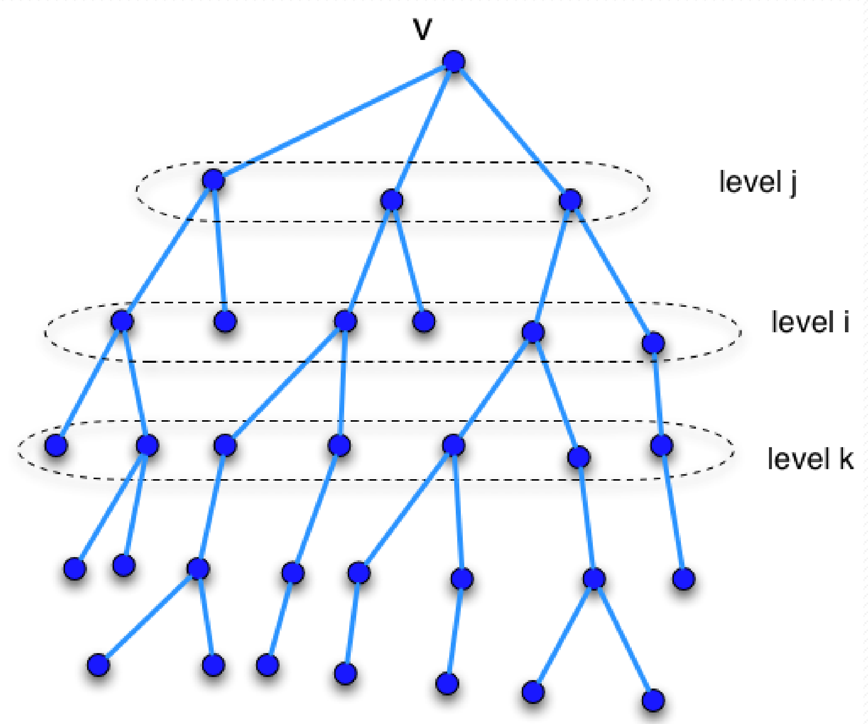
- Let G be a planar graph with nonnegative vertex costs whose sum ≤ 1
- Then the vertices of G can be partitioned into A,B,C
  - no edge joins A and B
  - neither A nor B has total cost $>2/3$
  - $|C| \leq 2\sqrt{2}n^{1/2}$
- We first assume G is connected

Proof

- Partition the vertices into levels by the shortest path tree from some vertex v
  - there is no edges links levels not adjacent to each other
  - Let L(l) is the number of vertices on level l
  - r is the number of levels, so we have level 0, level 1,…, level r

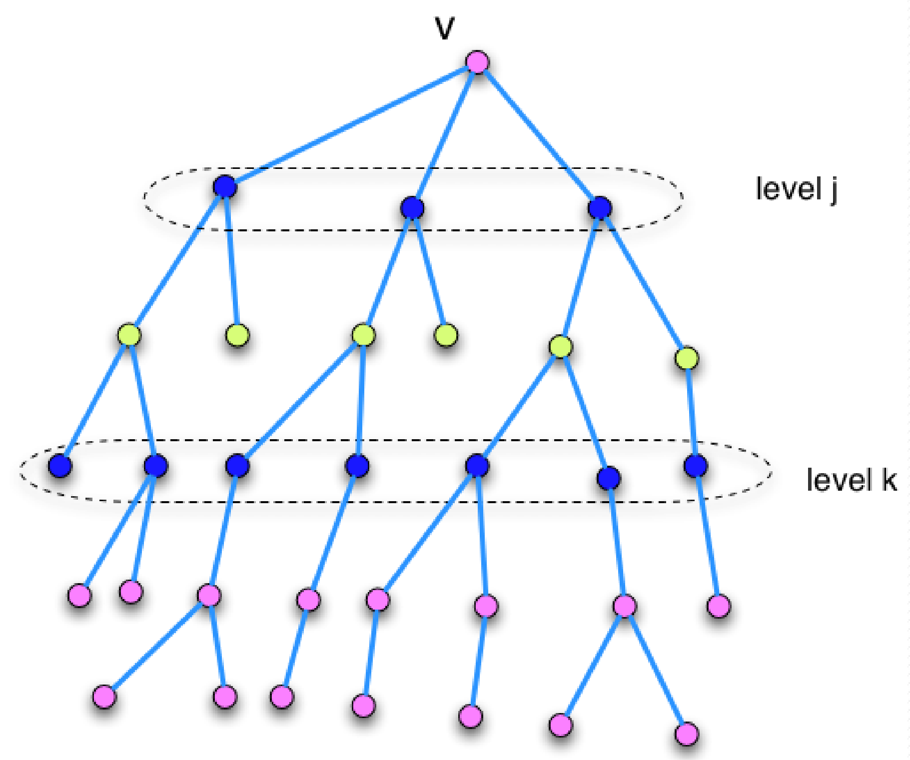


- Let i be the lowest level such that the total costs in level 0 to level i $\geq 1/2$,
  - denote the number of vertices in level 0 to level i by p
- Find $j \leq i$ and $k \geq i+1$ such that:
  - $|L(j)|+2(i-j) \leq 2p^{1/2}$
  - $|L(k)|+2(k-i-1) \leq 2(n-p)^{1/2}$

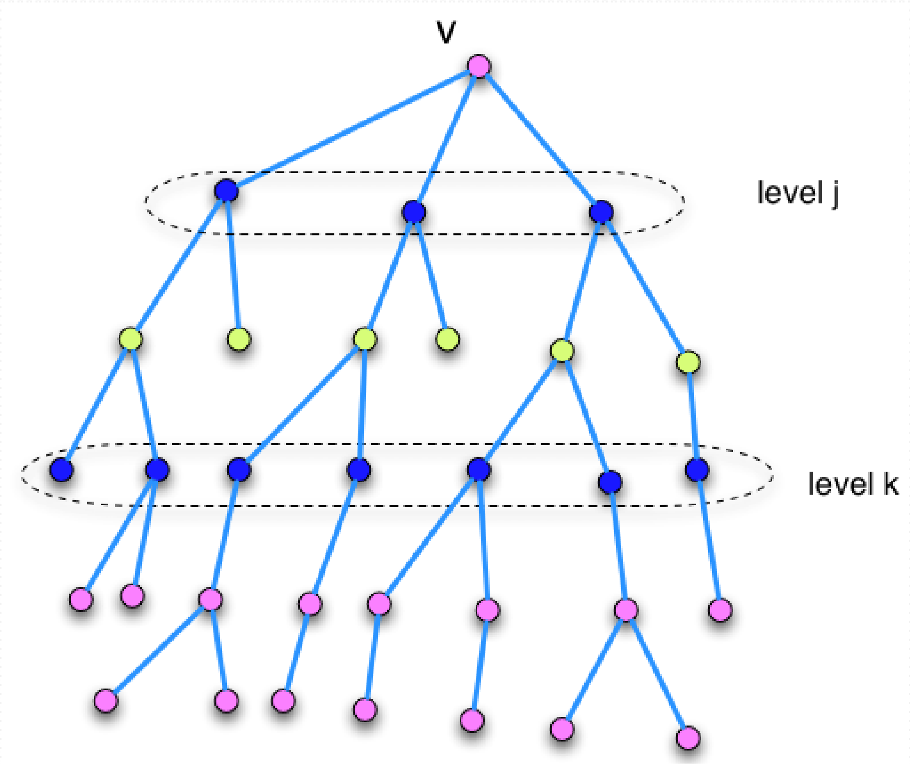


- Let i be the lowest level such that the total costs in level 0 to level i is $\geq 1/2$,
  - denote the number of vertices in level 0 to level i by p
- Find $j \leq i$ and $k \geq i+1$ such that:
  - $|L(j)|+2(i-j) \leq 2p^{1/2}$
  - $|L(k)|+2(k-i-1) \leq 2(n-p)^{1/2}$
- Then we will show that the conclusion follows by such j,k, and such j and k must exist

- Let i be the lowest level such that the total costs in level 0 to level i $\geq 1/2$,
  - denote the number of vertices in level 0 to level i by p
- Find $j \leq i$ and $k \geq i+1$ such that:
  - $|L(j)|+2(i-j) \leq 2p^{1/2}$
  - $|L(k)|+2(k-i-1) \leq 2(n-p)^{1/2}$
- Consider the vertices in levels: $[0,j-1]$, $[j+1,k-1]$,$[k+1,r]$.



- Consider the vertices in levels: [0,j-1], [j+1,k-1],[k+1,r].
- If the numbers of vertices in all of these sets are ≤2/3, then
  - C={vertices in levels j and k}, so $|C| \le 2\sqrt{2}n^{1/2}$
  - A=the biggest among these three sets
  - B=the union of the other two



$$|L(j)|+2(i-j) \le 2p^{1/2}$$
$$|L(k)|+2(k-i-1) \le 2(n-p)^{1/2}$$

- Consider the vertices in levels: $[0,j-1]$, $[j+1,k-1]$,$[k+1,r]$.
- If the number of vertices in one of these sets is $\leq 2/3$, then
  - C={vertices in levels j and k}, so $|C| \leq 2\sqrt{2}n^{1/2}$
- By definition of level i, j, k, the only sets which can has cost $>2/3$ is the middle part $[j+1,k-1]$
  - Delete all vertices in levels $[k,r]$
  - Shrink all vertices in levels $[0,j]$ to a single vertex with cost 0
  - This preserves planarity
- By our preliminary theorem, this tree T has a separator of size $2(i-j-1)+1$ vertices with one root.

If G has a spanning tree T of radius r, then G has a separator C, s.t. neither A nor B has total cost more than $2/3$, and C contains at most $2r+1$ vertices.

- Consider the vertices in levels: [0,j-1], [j+1,k-1],[k+1,r].
- If the number of vertices in one of these sets is ≤2/3, then
  - C={vertices in levels j and k}, so $|C| \leq 2\sqrt{2}n^{1/2}$
- By definition of level i, j, k, the only sets which can has cost >2/3 is the middle part [j+1,k-1]
  - Delete all vertices in levels [k,r]
  - Shrink all vertices in levels [0,j] to a single vertex with cost 0
  - This preserves planarity
- By our preliminary theorem, this tree T has a separator of size 2(i-j-1)+1 vertices with one root.
  - So this separator with L(j) and L(k) will form a separator of size $|L(j)|+|L(k)|+2(i-j-1) \leq 2\sqrt{2}n^{1/2}$

$|L(j)|+2(i-j) \leq 2p^{1/2}$
$|L(k)|+2(k-i-1) \leq 2(n-p)^{1/2}$

- Let i be the lowest level such that the total costs in level 0 to level i is $\geq 1/2$,
  - denote the number of vertices in level 0 to level i by p
- Find $j \leq i$ and $k \geq i+1$ such that:
  - $|L(j)|+2(i-j) \leq 2p^{1/2}$
  - $|L(k)|+2(k-i-1) \leq 2(n-p)^{1/2}$
- Then we will show that the conclusion follows by such j,k, and such j and k must exist

- Let i be the lowest level such that the total costs in level 0 to level i is $\geq 1/2$,
  - denote the number of vertices in level 0 to level i by p
- Find $j \leq i$ and $k \geq i+1$ such that:
  - $|L(j)|+2(i-j) \leq 2p^{1/2}$
  - $|L(k)|+2(k-i-1) \leq 2(n-p)^{1/2}$
- If such j does not exists, then for all $h \leq j$, $L(h) > 2p^{1/2}-2(i-h)$, and $L(0)=1$, so $i+1/2 \geq p^{1/2}$, so

$$p = \sum_{h=0}^{i} L(h) \geq \sum_{h=0}^{i} 2\sqrt{p} - 2(i-h) > p$$

- We can prove k exists by a similar procedure.

Main theorem:

- Let G be a planar graph with nonnegative vertex costs whose sum ≤ 1
- Then the vertices of G can be partitioned into A,B,C
  - no edge joins A and B
  - neither A nor B has total cost $>2/3$
  - $|C| \leq 2\sqrt{2}n^{1/2}$

- We first assume G is connected
- Otherwise, For each connected component, we can find a separator.

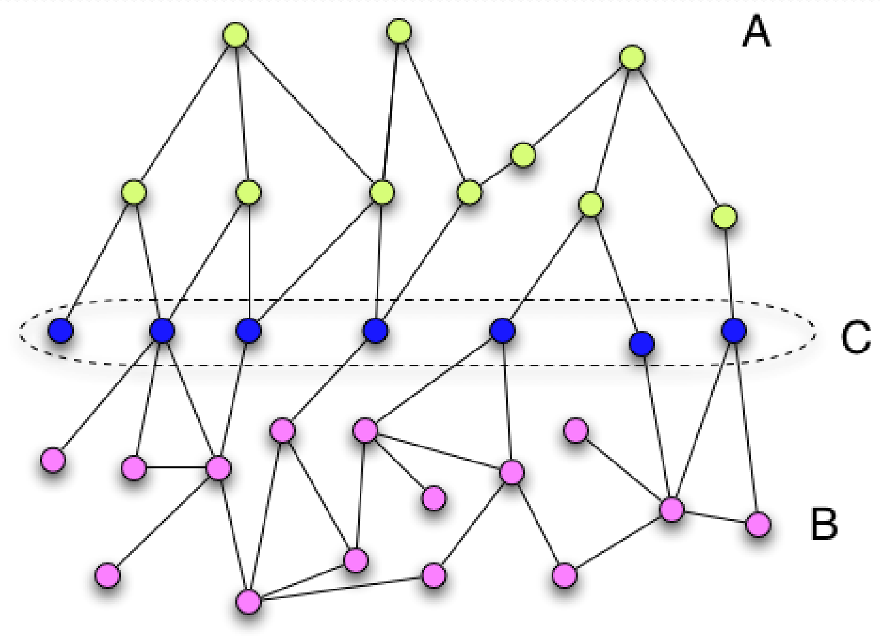
Simpler version:

- Then the vertices of G can be partitioned into A,B,C
  - no edge joins A and B
  - $|A|, |B| \leq \frac{2}{3}n$
  - $|C| \leq 2\sqrt{2}n^{1/2}$

  - By assign each vertex of G a cost of 1/n.

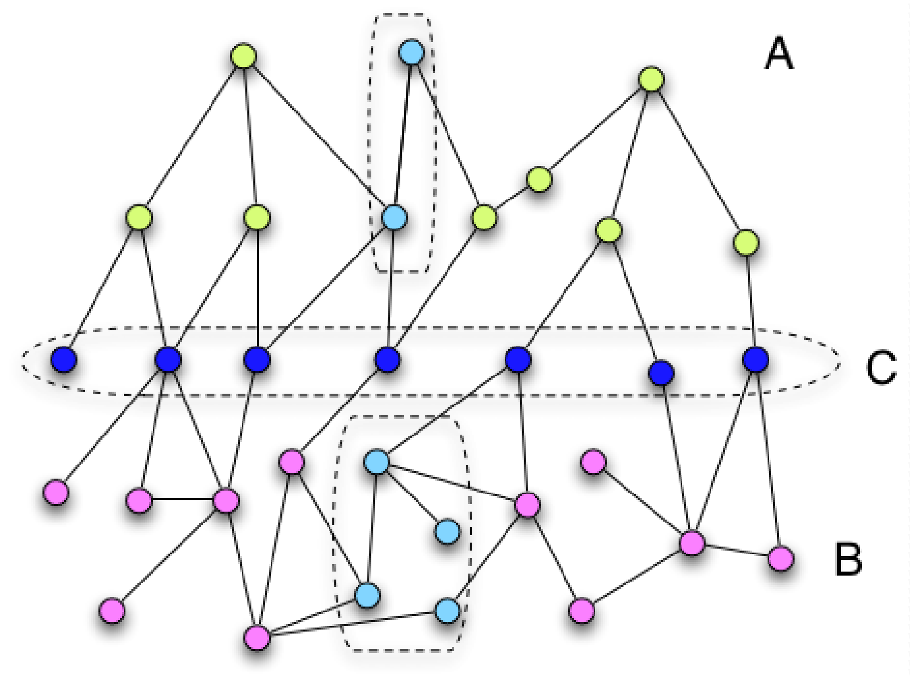
Applications

- Very useful in divide-and-conquer method
- For A and B, recursively find separators in them



Applications

- Very useful in divide-and-conquer method
- For A and B, recursively find separators in them



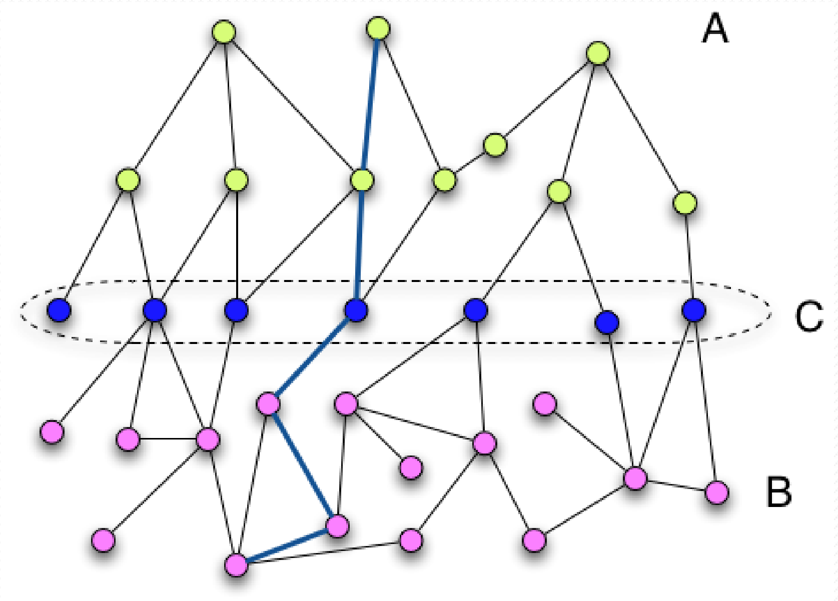
Example

- Data structure to store all-pair shortest paths
- n×n table

	v_1	v_2	…	v_n
v_1				
v_2				
…				
v_n				

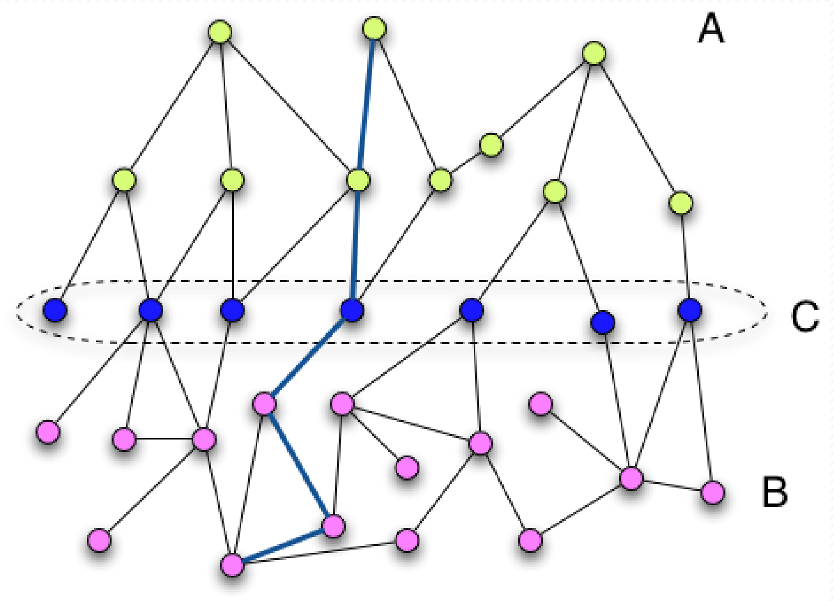
Distance structure for planar graph

- The path between A and B must go through the separator C



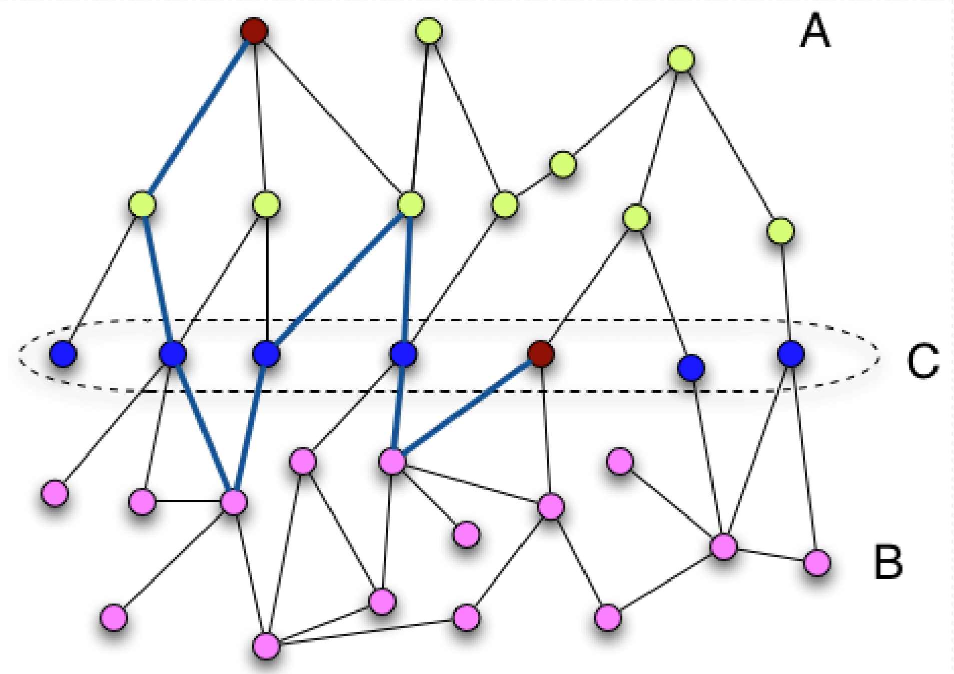
Distance structure for planar graph

- The path between A and B must go through the separator C
- So we store the distance of (u,v) where u∈C and v∈V
  - Space: $O(n^{3/2})$



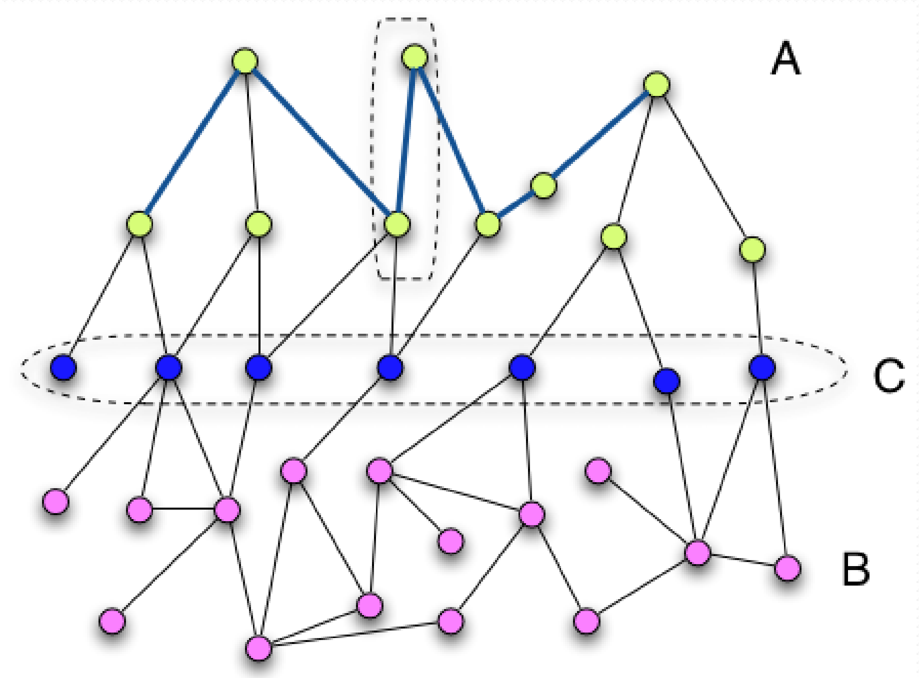
Distance structure for planar graph

- The path between A and B must go through the separator C
- So we store the distance of d(u,v) where u∈C and v∈V
  - Space: $O(n^{3/2})$
  - Note that the path between A and C may travel through B



Distance structure for planar graph

- The path between A and B must go through the separator C
- So we store the distance of (u,v) where u∈C and v∈V
  - Space: $O(n^{3/2})$
- For each of A and B, recursively construct the structure.

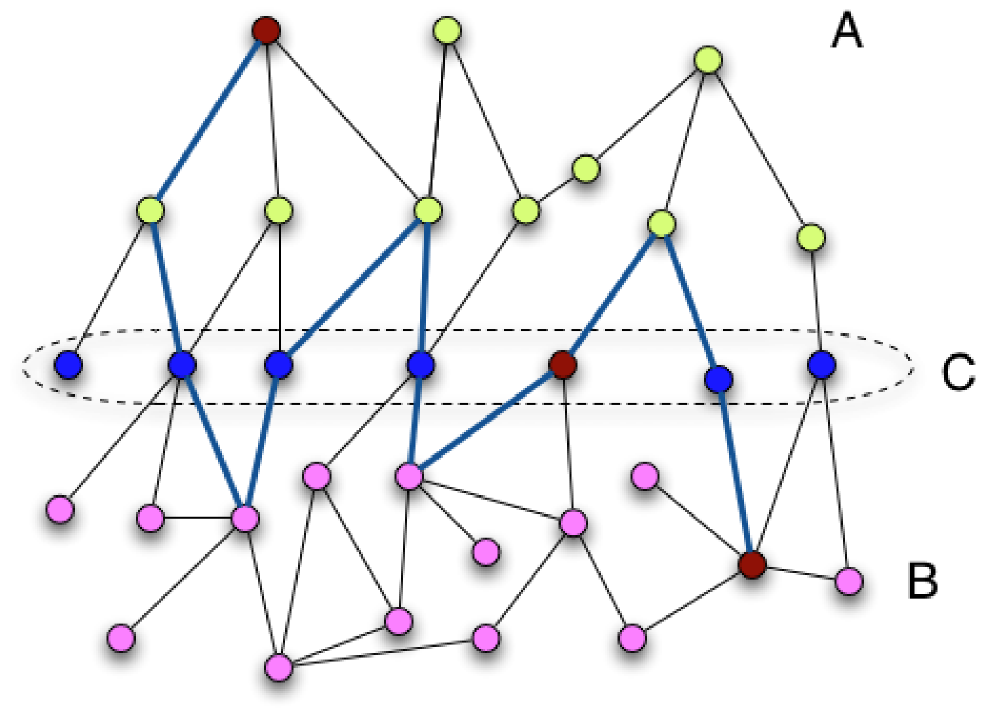


Distance structure for planar graph

- The path between A and B must go through the separator C
- So we store the distance of (u,v) where u∈C and v∈V
  - Space: $O(n^{3/2})$
- For each of A and B, recursively construct the structure.
  - For A, store the distances between all vertices of A and the vertices of separator of A, where the paths only travel within A
- Total space: $O\left(n^{3/2} + 2\left(\frac{n}{2}\right)^{3/2} + 4\left(\frac{n}{4}\right)^{3/2} + \ldots\right) = O(n^{3/2})$

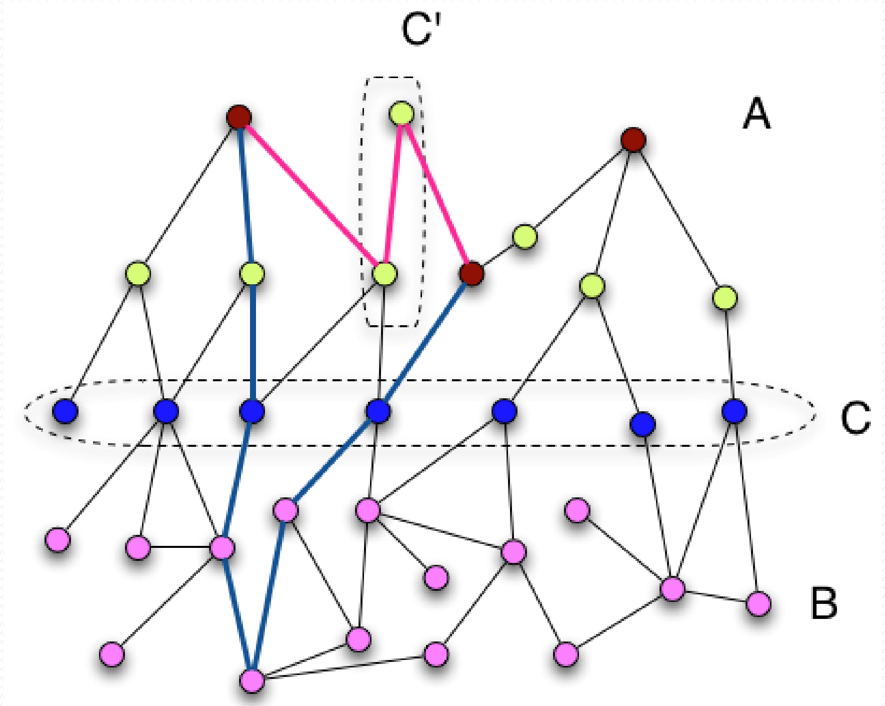
Answering a query (u,v)

- If $u \in A$ and $v \in B$, then just find the minimum of:
  - $\min\{d(u,w)+d(w,v) \mid w \in C\}$
  - Query time: $O(n^{1/2})$



Answering a query (u,v)

- If both u,v∈A but in different subparts, find the minimum of:
  - min{d(u,w)+d(w,v)|w∈C}, min{d(u,w')+d(w',v)|w'∈C'}
  - This will cover:
    - paths travels through C, paths within A

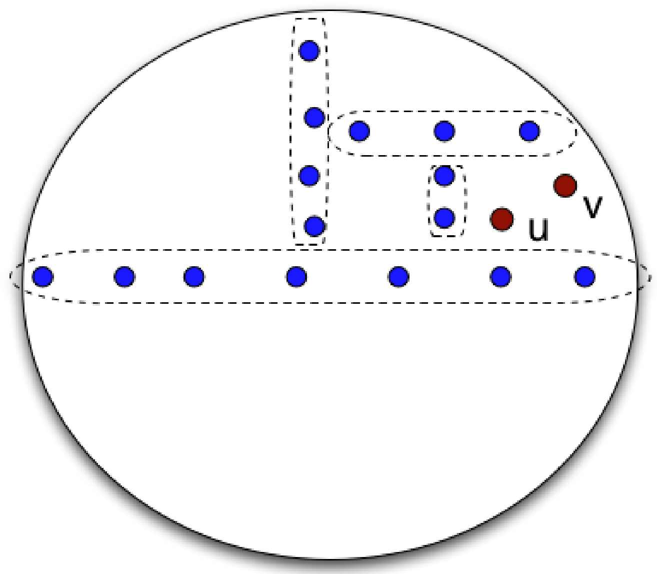


Answering a query (u,v)

- If both $u,v \in A$ but in different subparts, find the minimum of:
  - $\min\{d(u,w)+d(w,v) \mid w \in C\}$, $\min\{d(u,w')+d(w',v) \mid w' \in C'\}$
  - This will cover:
    - paths travels through C, paths within A
  - Query time: $O(n^{1/2})$

Answering a query (u,v)

- Thus, for any u,v in some subpartition, we need to check the vertices on the borders:
- Query time: $$O\left(\sqrt{n}+\sqrt{\frac{n}{2}}+\sqrt{\frac{n}{4}}+\ldots\right)=O(\sqrt{n})$$



Distance structure for planar graphs

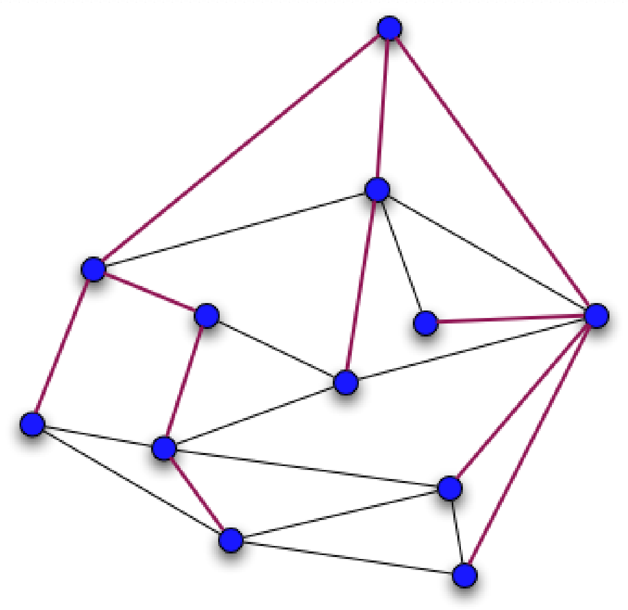
- Instead of store an $O(n^2)$ table, we can construct a structure of space $O(n^{3/2})$ with query time $O(n^{1/2})$.

- So most problems in graph theory have faster algorithms for planar graphs than for general graphs.

- So most problems in graph theory have faster algorithms for planar graphs than for general graphs.
- $O(n^{1/2})$-separator is more common, but the path separator is also useful:
  - *"Compact oracles for reachability and approximate distances in planar digraphs"*
  - Mikkel Thorup, 2004

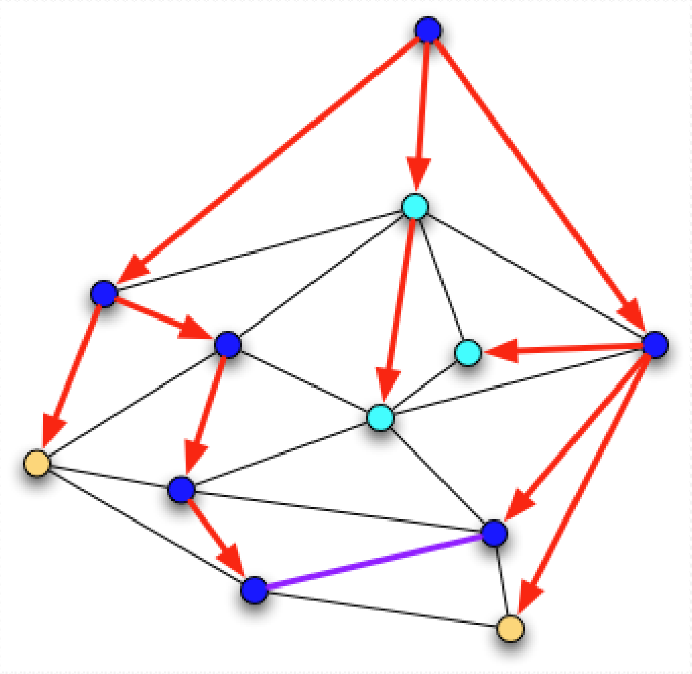
Preliminary theorem:

- Let G be a planar graph with nonnegative vertex costs whose sum ≤1
- If G has a spanning tree T of radius r, then G has a separator C, s.t. neither A nor B has total cost more than 2/3, and C contains at most 2r+1 vertices.



Preliminary theorem:

- If G has a spanning tree T of radius r, then G has a separator C, s.t. neither A nor B has total cost more than 2/3, and C contains at most 2r+1 vertices.
- C is a cycle formed by the tree and a non-tree edge.
- If the tree is a shortest path tree from or to the root, the separator C is on two paths.



Thank you!