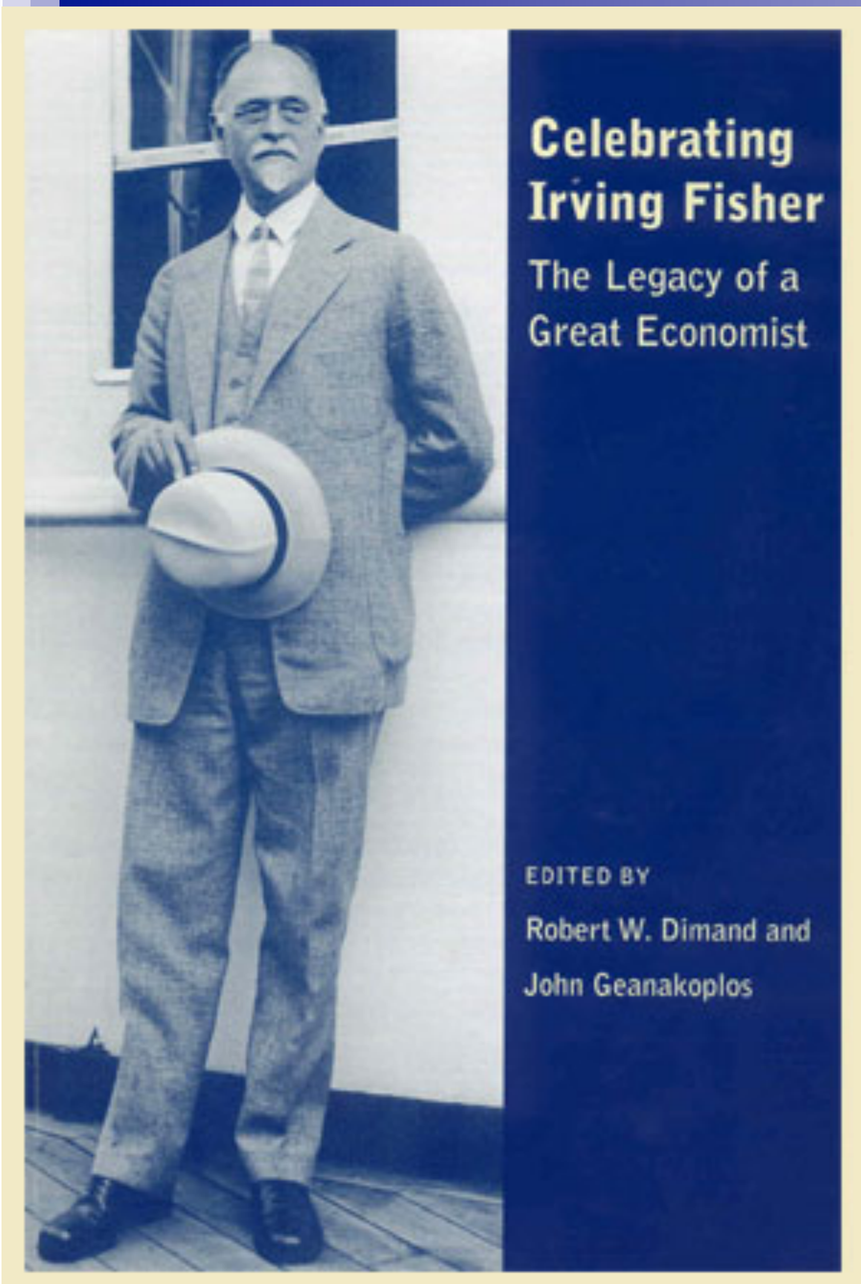
Combinatorial Algorithms for Linear Fisher Markets

In this talk

- Fisher market model and Arrow-Debreu market model
- Combinatorial algorithm solving linear Fisher market
  - Similar to matching and maximum flow

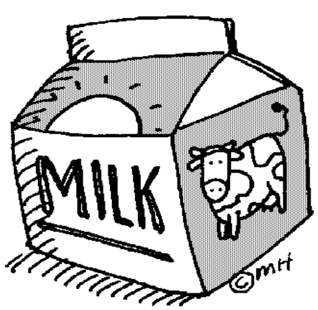
- Not in exam

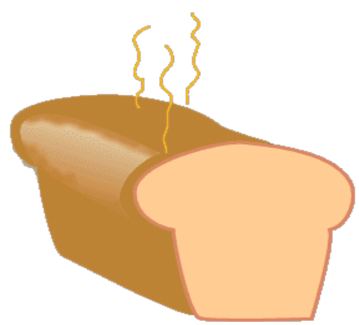


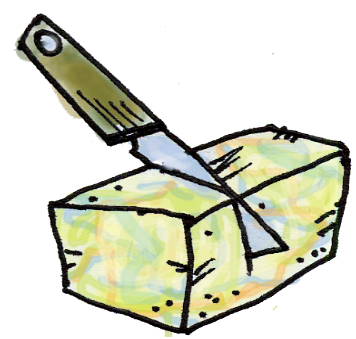
Irving Fisher, 1891

- Defined a fundamental market model
- Special case of Walras' model

Several buyers with different utility functions and moneys.

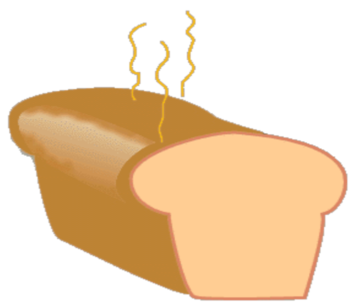




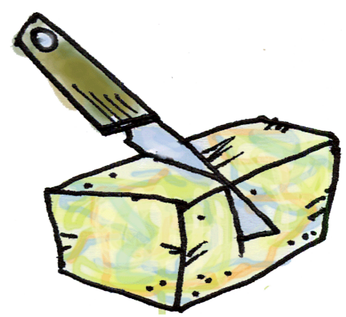


Several buyers with different utility functions and moneys. Find equilibrium prices.

p_1



p_2



p_3

Linear Fisher Market

- $B = n$ buyers, money m_i for buyer i
- $G = g$ goods, w.l.o.g. unit amount of each good

Linear Fisher Market

- $B = n$ buyers, money m_i for buyer i
- $G = g$ goods, w.l.o.g. unit amount of each goods
- u_{ij} : utility derived by i
  on obtaining one unit of j
  - *Desirability*

Linear Fisher Market

- $B = n$ buyers, money m_i for buyer i
- $G = g$ goods, w.l.o.g. unit amount of each good
- u_{ij} : utility derived by i

  on obtaining one unit of j
- Total utility of i,

$$v_i = \sum_j u_{ij} x_{ij}$$

$$x_{ij} \in [0,1]$$

Linear Fisher Market

- $B = n$ buyers, money m_i for buyer i
- $G = g$ goods, w.l.o.g. unit amount of each good
- u_{ij} : utility derived by i

  on obtaining one unit of j
- Total utility of i,

$$v_i = \sum_j u_{ij} x_{ij}$$

$$x_{ij} \in [0,1]$$

- Find market clearing prices.

Market clearing allocation

- Budget constraint
  - $\sum_j p_j x_{ij} \le m_i$
  - The buyer cannot spend more than what he has
- Optimality
  - x maximize $\sum_j u_{ij} x_{ij}$ for every buyer i
  - no other bundle of goods that satisfies the budget constraint is more desirable
- Market clearing
  - There is neither deficiency nor surplus of any goods:

    $$\forall j, \sum_i x_{ij} = 1$$

People | Goods

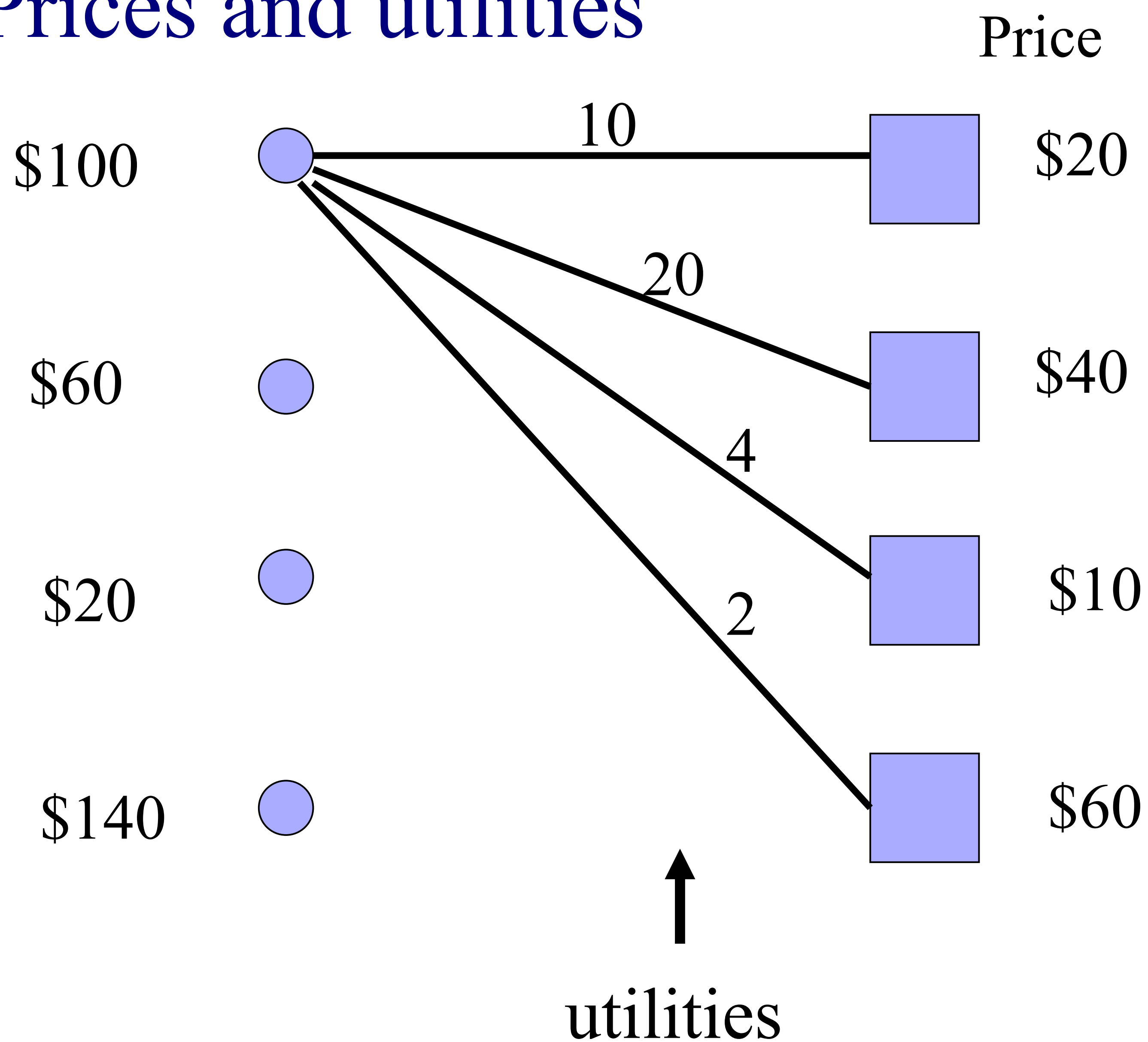
$100

$60

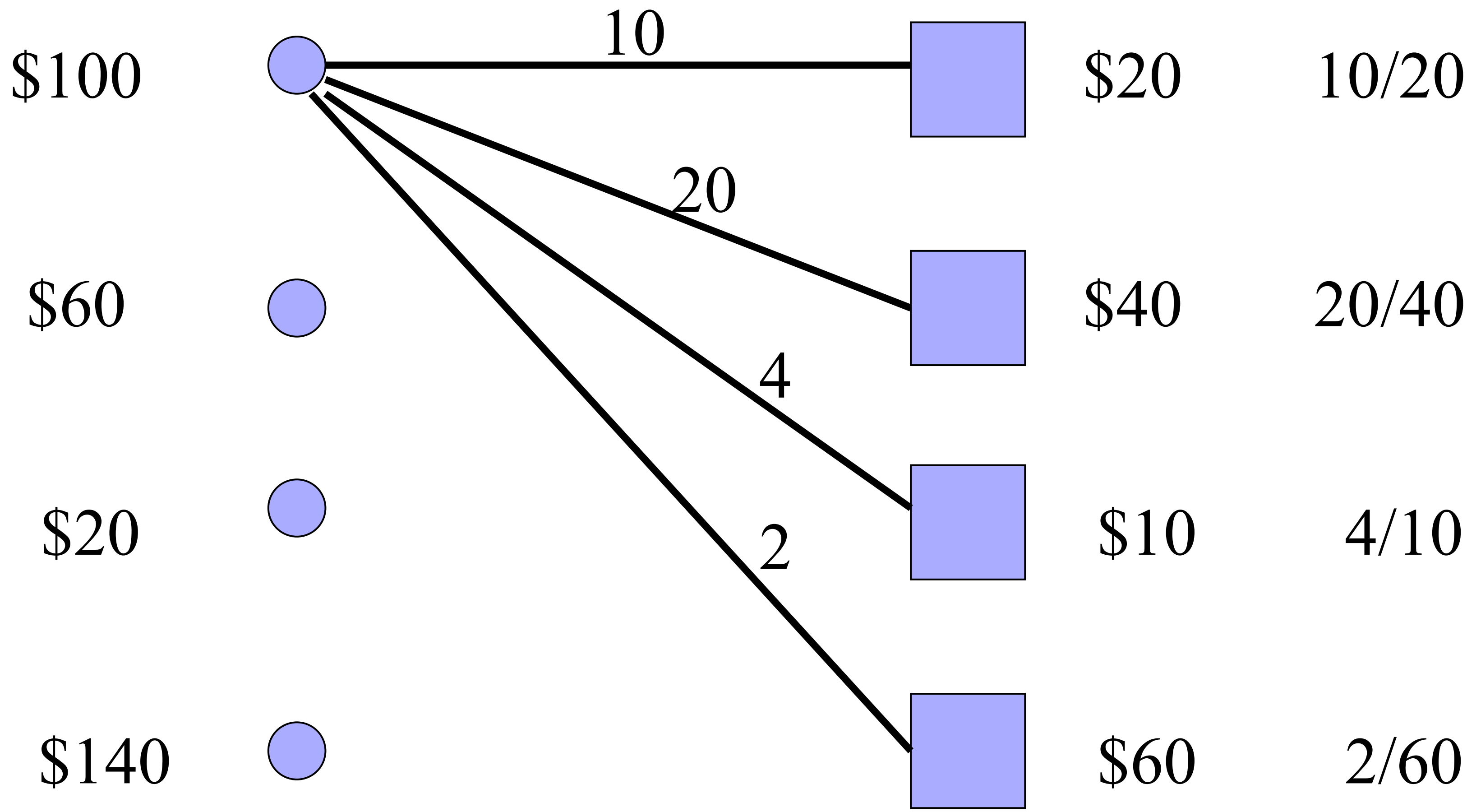
$20

$140

Prices and utilities



Bang per buck



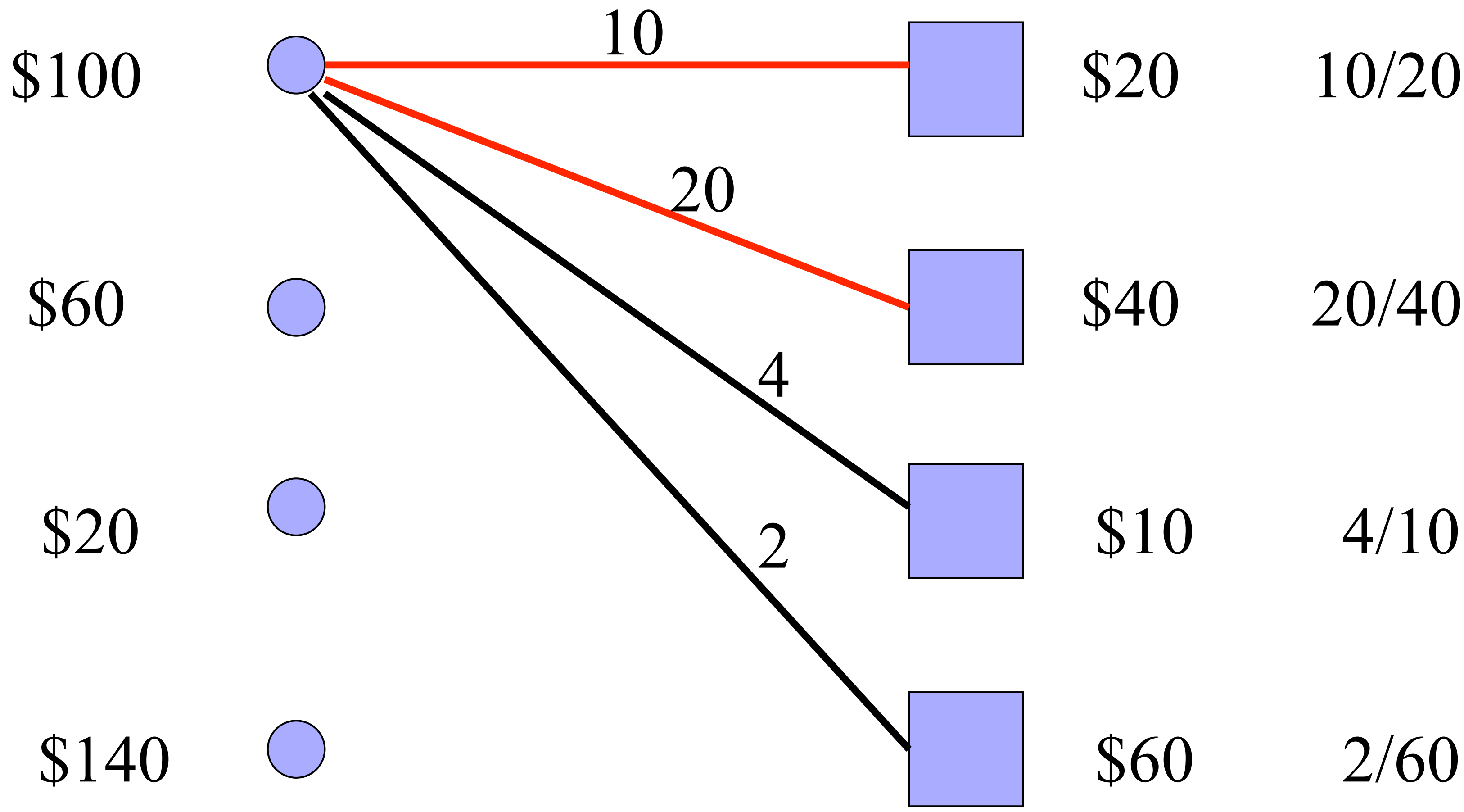
Bang per buck

- Utility of $1 worth of goods

- Buyers will only buy goods providing maximum bang per buck

$$\max_j \frac{u_{ij}}{p_j}$$

Equality subgraph



$100 can get utility of 50

Arrow-Debreu market

- Each agent is assigned a bundle of goods instead of an amount of money
- They sell their own goods to other agents then buy their desirable goods
- Fisher market can be seen as a special case of AD market
  - Money can be seen as a kind of goods

Arrow-Debreu Theorem, 1954

- Established existence of market equilibrium under very general conditions using a deep theorem from topology - Kakutani fixed point theorem.

- Provides a mathematical ratification of Adam Smith's "invisible hand of the market".

Need algorithmic ratification!!

Linear Fisher Market

- $B = n$ buyers, money m_i for buyer i
- $G = g$ goods, w.l.o.g. unit amount of each good
- u_{ij} : utility derived by i

  on obtaining one unit of j
- Total utility of i,

$$v_i = \sum_j u_{ij} x_{ij}$$

$$x_{ij} \in [0,1]$$

- Find market clearing prices.

Can equilibrium allocations be captured via an LP?

- Set of feasible allocations:

$$\forall j \quad \sum_i x_{ij} \le 1$$

$$\forall i, j \quad x_{ij} \ge 0$$

Does equilibrium optimize a global objective function?

- Guess 1: Maximize sum of utilities, i.e.,

$$\max \sum_i u_i(x) = \max \sum_i \sum_j u_{ij} x_{ij}$$

- Problem: $u_i(x)$ and $2 \times u_i(x)$

  are equivalent utility functions.

However,

Maximize $u_1(x) + \sum_{i \neq 1} u_i(x)$ does not necessarily

maximize $2u_1(x) + \sum_{i \neq 1} u_i(x)$

$\forall j \quad \sum_i x_{ij} \leq 1$

$\forall i, j \quad x_{ij} \geq 0$

Guess 2: Product of utilities.

$$\text{Maximize} \quad u_1(x) \times \prod_{i \neq 1} u_i(x)$$

$$\text{maximizes} \quad 2u_1(x) \times \prod_{i \neq 1} u_i(x)$$

$$\forall j \quad \sum_i x_{ij} \leq 1$$

$$\forall i, j \quad x_{ij} \geq 0$$

- However, suppose a buyer with \$200 is split into two buyers with \$100 each And same utility function.

- Clearly, equilibrium should not change.

However,

Maximize $u_1(x) \times \prod_{i \neq 1} u_i(x)$ does not necessarily

maximize $u_1(x)^2 \times \prod_{i \neq 1} u_i(x)$

$\forall j \quad \sum_i x_{ij} \leq 1$

$\forall i, j \quad x_{ij} \geq 0$

- Money of buyers is relevant.
- Assume a utility function is written on each dollar in market

Guess 3: Product of utilities over all dollars

$$\text{Max} \prod_i u_i(x)^{m(i)}$$

$$\forall j \quad \sum_i x_{ij} \leq 1$$

$$\forall i, j \quad x_{ij} \geq 0$$

Eisenberg-Gale Program, 1959

$$\max \sum_i m(i) \log u_i$$

$$s.t.$$

$$\forall i : u_i = \sum\nolimits_j u_{ij} x_{ij}$$

$$\forall j : \sum\nolimits_i x_{ij} \leq 1$$

$$\forall ij : x_{ij} \geq 0$$

This can be solved in polynomial time
However, we want combinatorial algorithms

An easier question

- Given prices $\underline{p},$ are they equilibrium prices?
- If so, find equilibrium allocations.

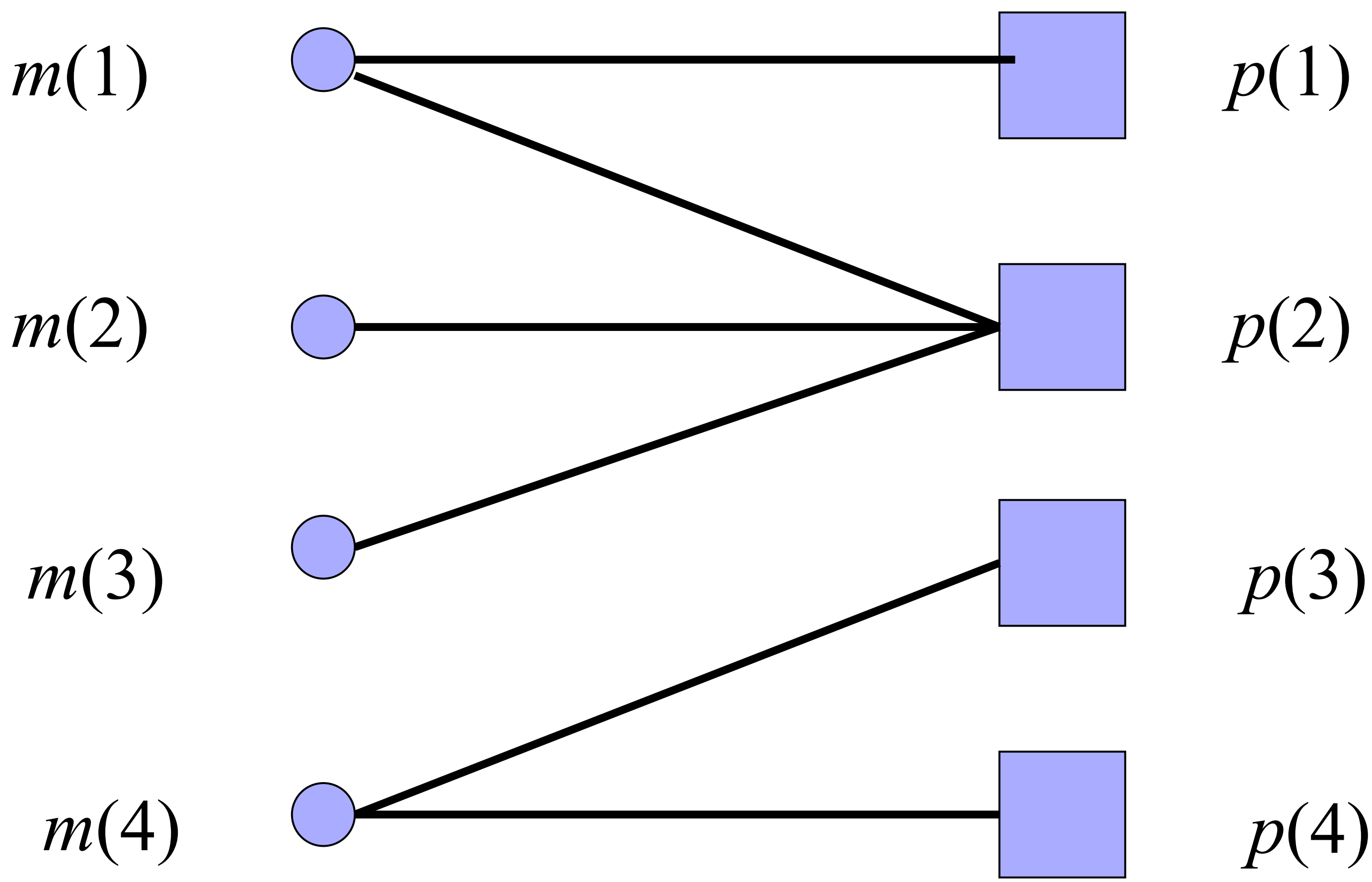
Bang-per-buck

- At prices $\underline{p}$, buyer i's most desirable goods, $$S_i = \arg\max_j \frac{u_{ij}}{p_j}$$

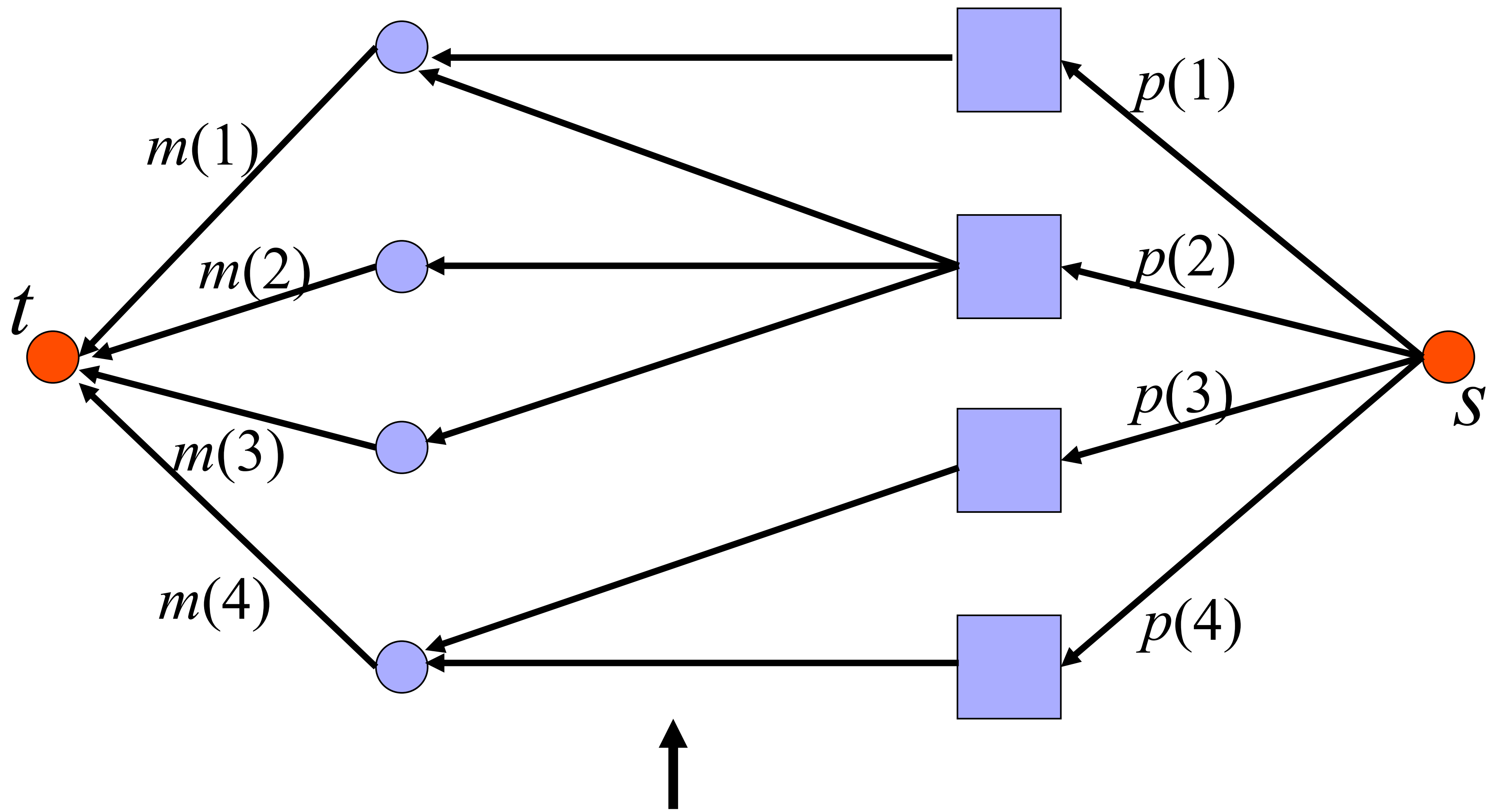
- Any goods from S_i worth $m(i)$ constitute i's optimal bundle

For each buyer, most desirable goods, i.e.

$$S_i = \arg\max_j \left\{ \frac{u_{ij}}{p_j} \right\}$$

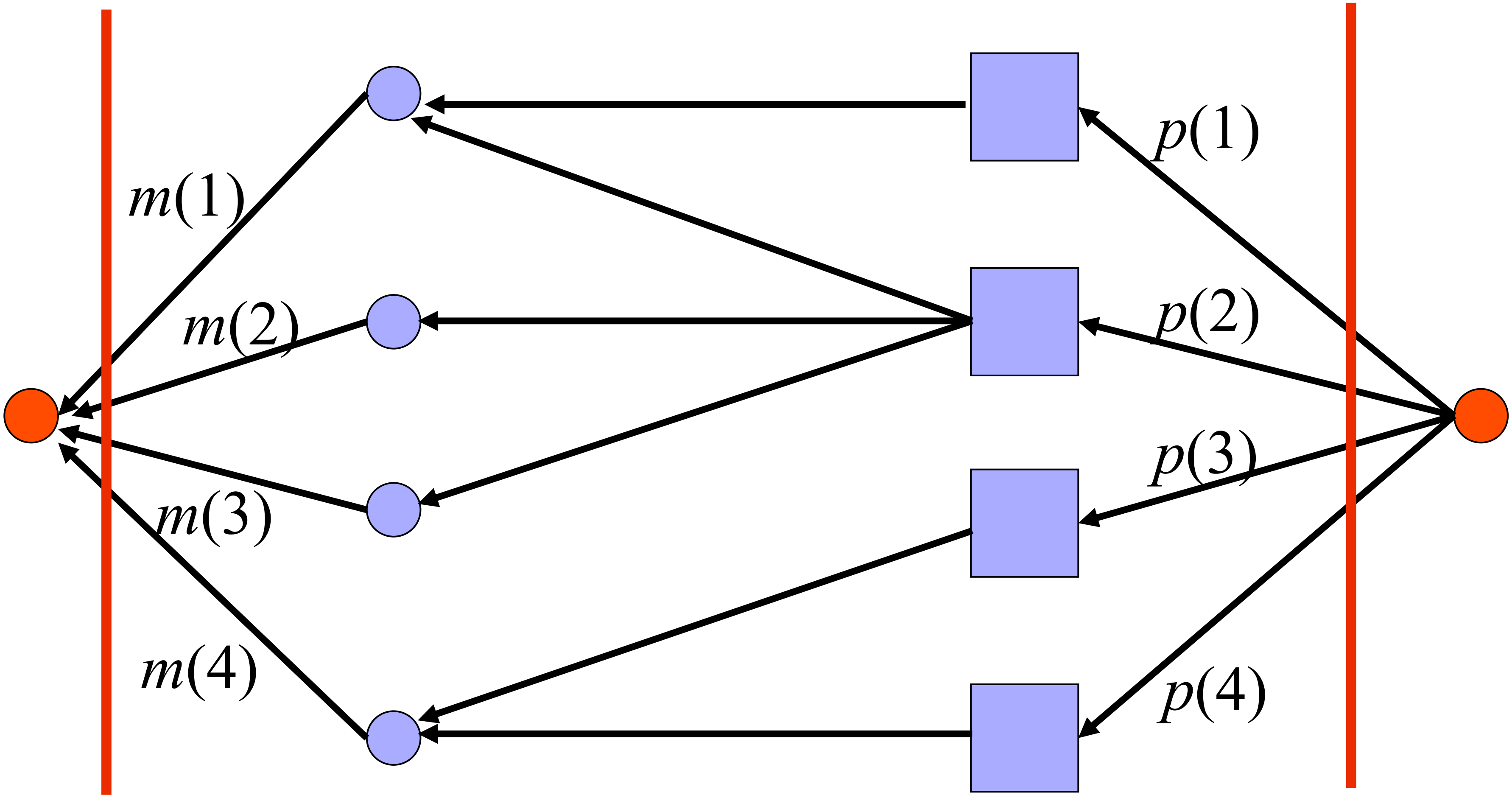


Network $N(p)$



infinite capacities

Max flow in *N(p)*



p: equilibrium prices iff both cuts saturated

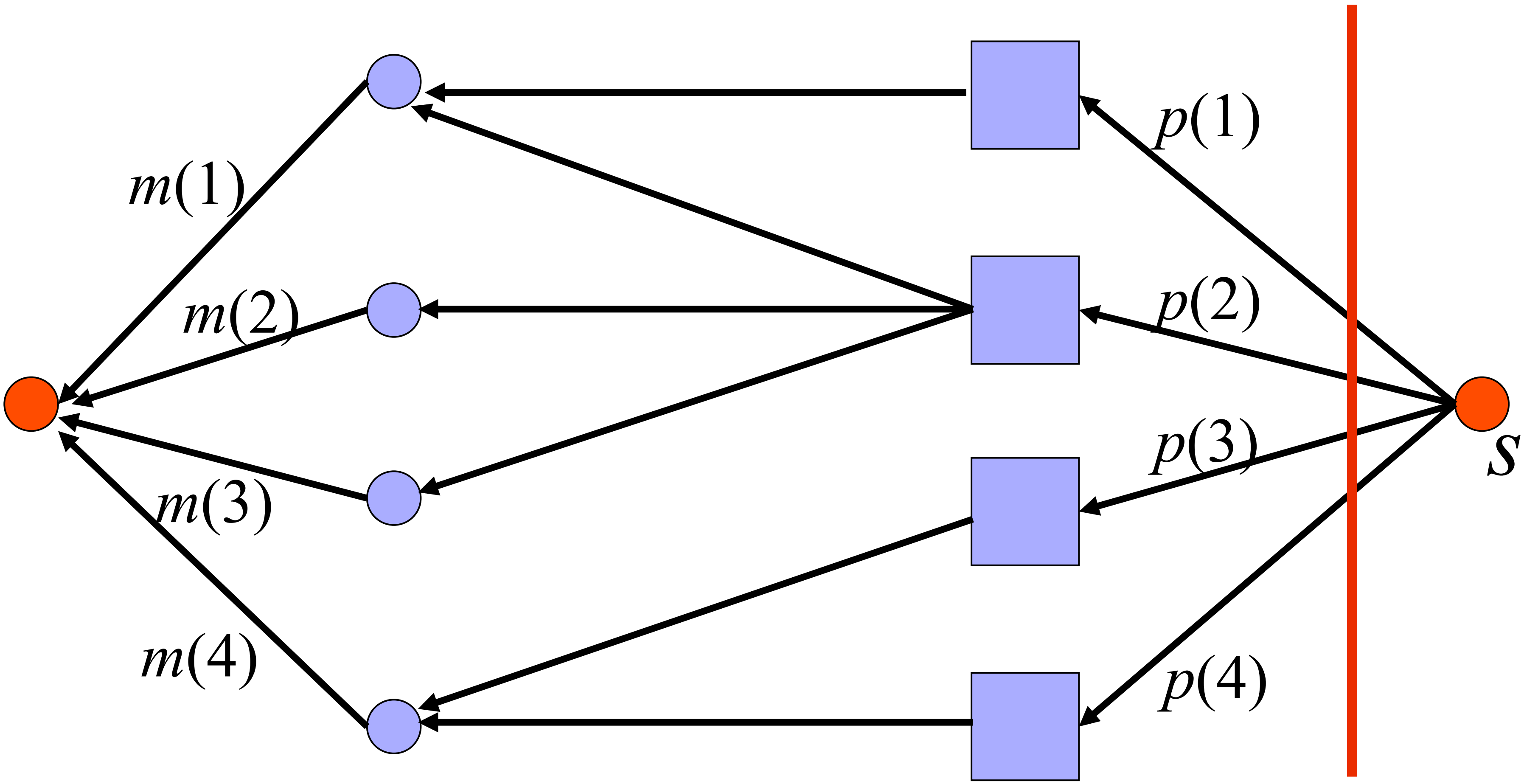
Idea of algorithm

- "primal" variables: allocations
- "dual" variables: prices of goods
- Approach equilibrium prices from below:
  - start with very low prices; buyers have surplus money
  - iteratively keep raising prices
    and decreasing surplus

An important consideration

- The price of a good never exceeds its equilibrium price
  - Invariant: s is a min-cut

Invariant: s is a min-cut in $N(\underline{p})$



$\underline{p}$: low prices

Idea of algorithm

- Iterations:

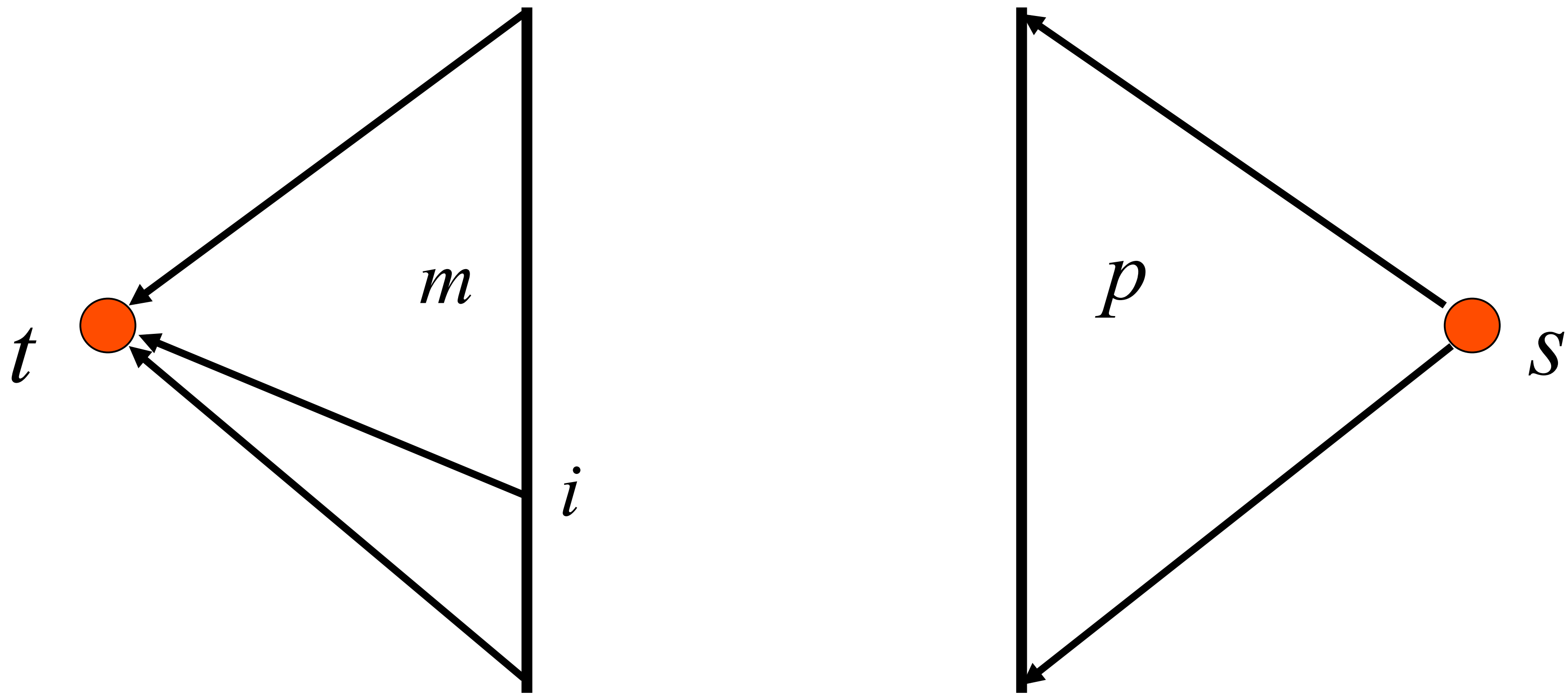
  execute primal & dual improvements

$$\begin{tikzcd}
\text{Allocations} \arrow[r, bend right] & \text{Prices} \arrow[l, bend right]
\end{tikzcd}$$

Key Algorithmic Idea

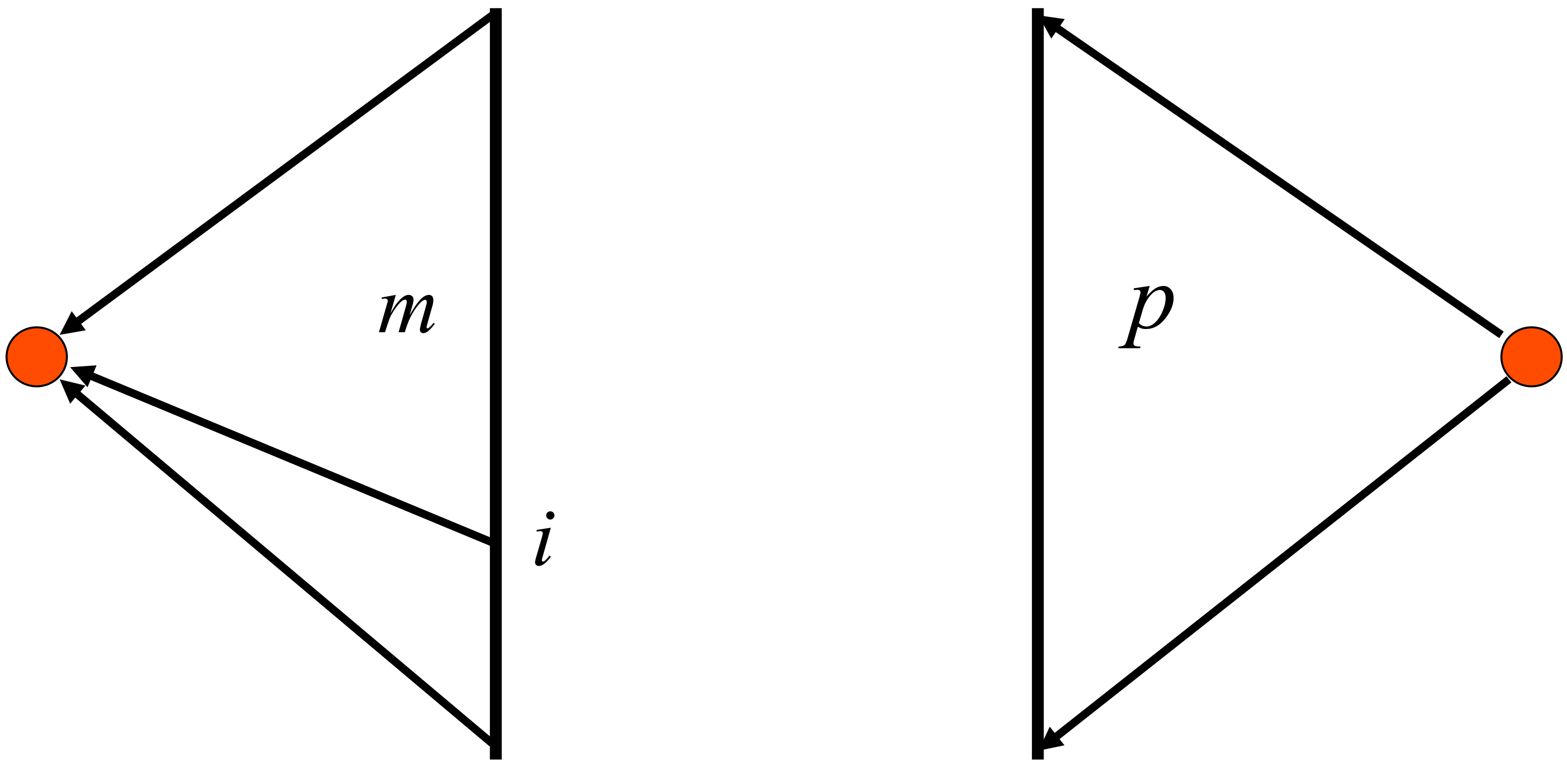
- Dual variables (prices) are raised greedily
- Yet, primal objects go tight and loose
- Balanced Flows: For limiting no. of such events

Max-flow in N



W.r.t. max-flow f, surplus(i) $= m(i) - f(i,t)$

Max-flow in N



surplus vector $=$ vector of surpluses w.r.t. f

Obvious potential function

- Total surplus money $=$ l_1 norm of surplus vector
- Reduce l_1 norm of surplus vector by inverse polynomial fraction in each iteration ?

$$l_1(s_1, s_2, \ldots, s_n) = |s_1| + |s_2| + \ldots + |s_n|$$

Balanced flow

- A max-flow that minimizes l_2 norm of surplus vector.

- Makes surpluses as equal as possible.

$$l_2(s_1, s_2, \ldots, s_n) = \sqrt{s_1^{\,2} + s_2^{\,2} + \ldots + s_n^{\,2}}$$

Balanced flow

- A max-flow that
  minimizes l_2 norm of surplus vector.

- Makes surpluses as equal as possible.

- All balanced flows have same surplus vector.

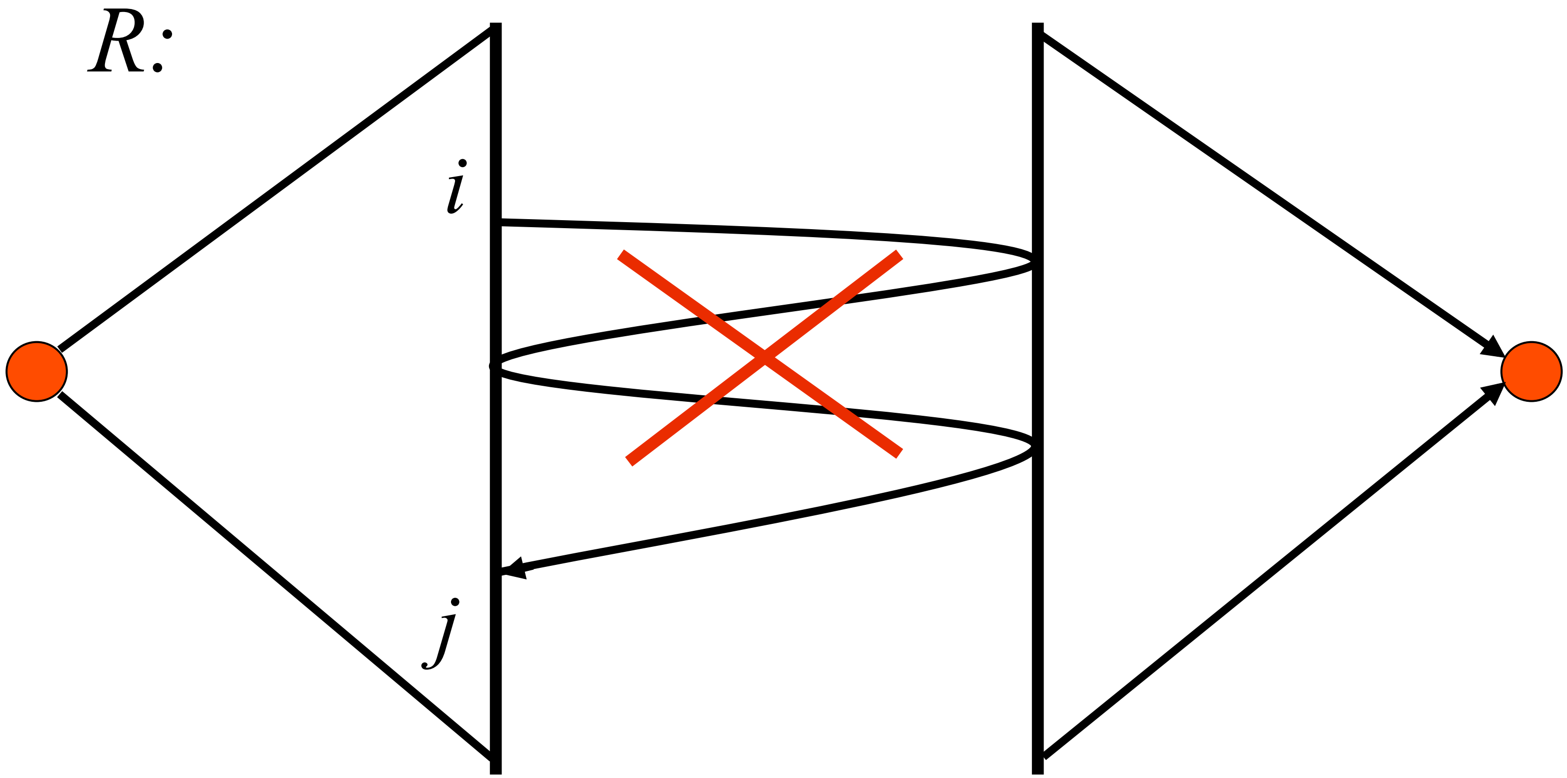
Our algorithm

- Reduces l_2 norm of surplus vector by inverse polynomial fraction in each iteration.

Property 1

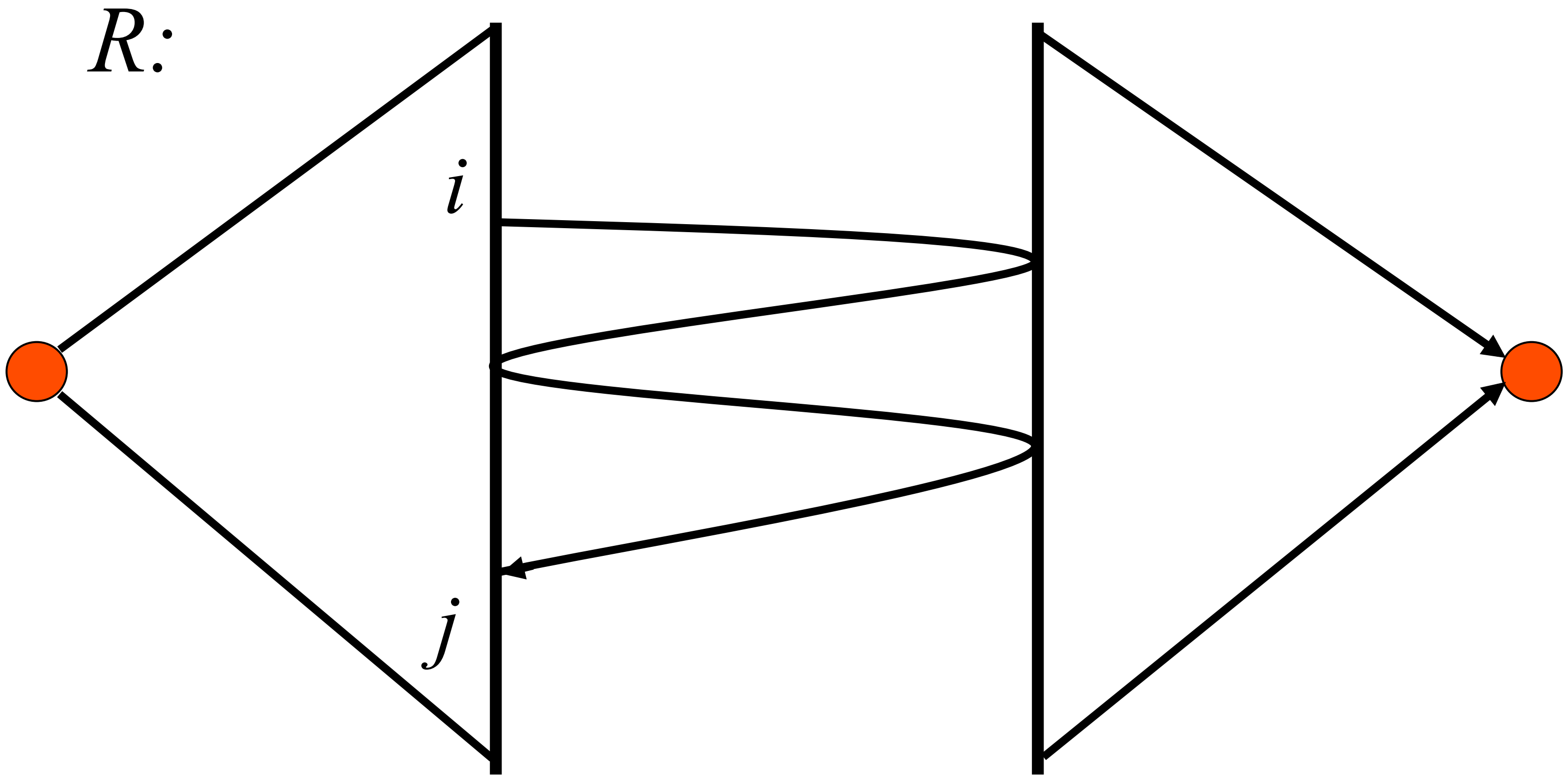
- f: max-flow in N.
- R: residual graph w.r.t. f.
- If surplus (i) $<$ surplus(j) then there is no path from i to j in R.

Property 1



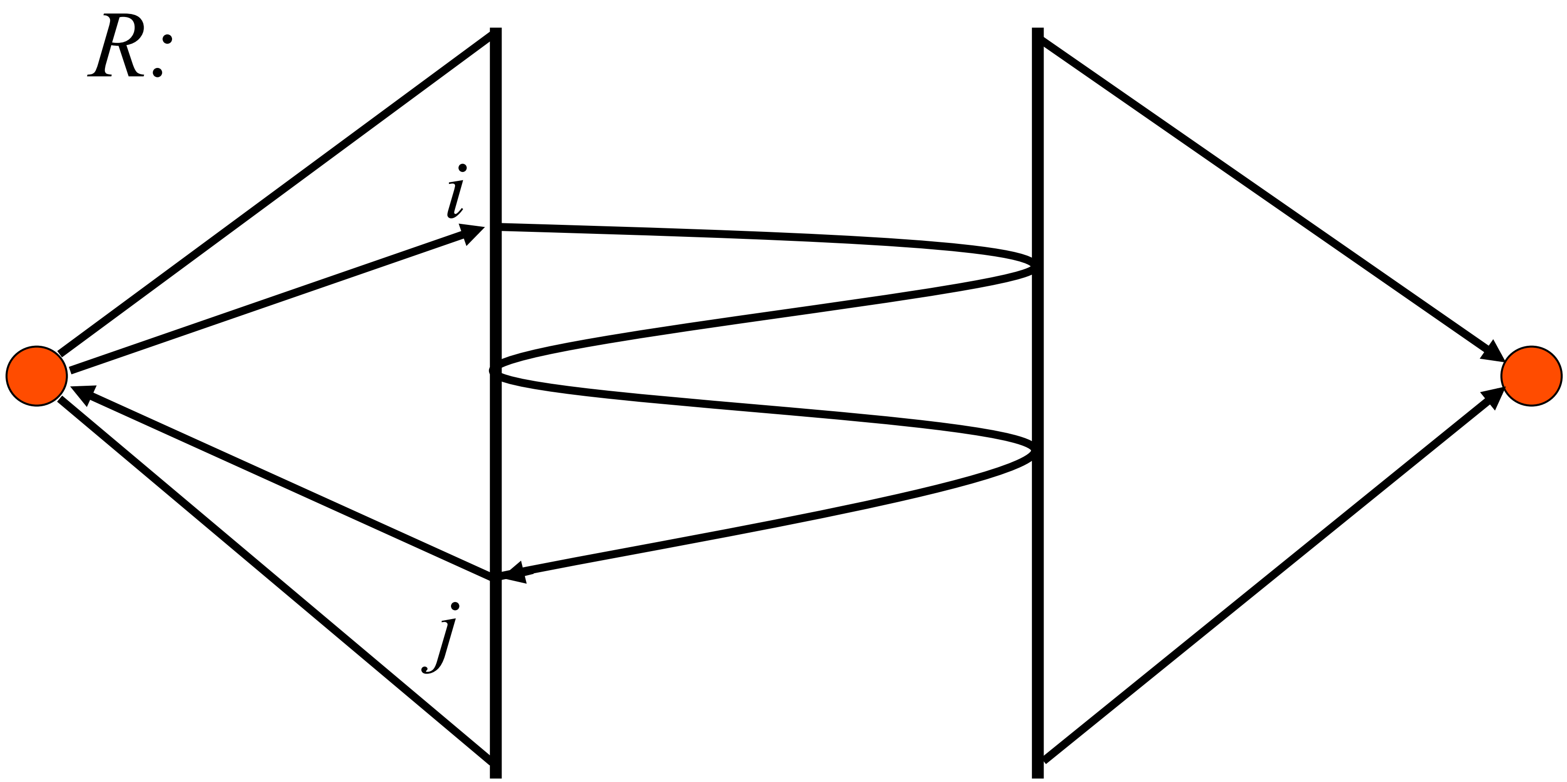
surplus(i) < surplus(j)

Property 1



surplus(i) < surplus(j)

Property 1



Circulation gives a more balanced flow.

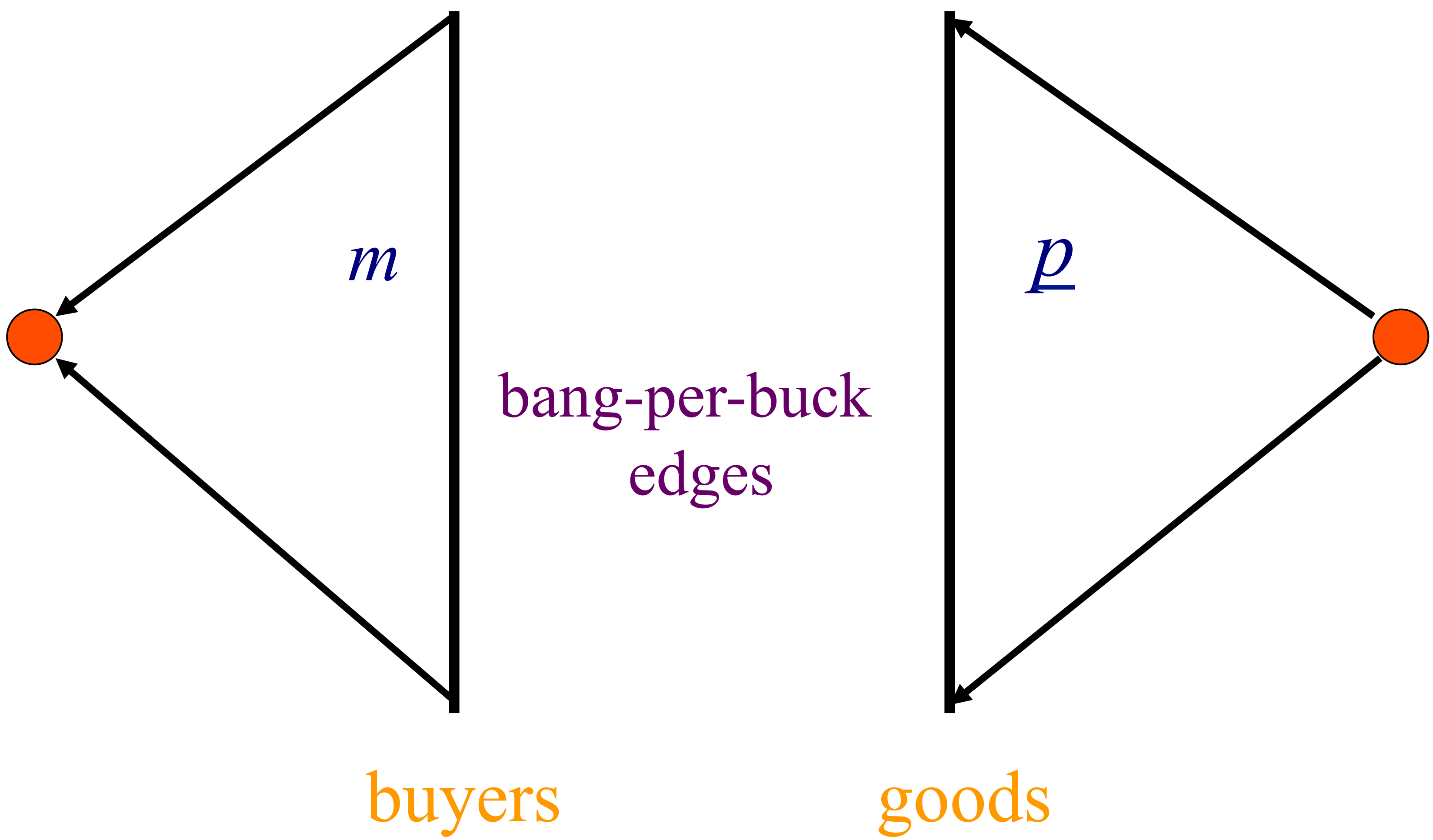
Property 1

- Theorem: A max-flow is balanced iff
  it satisfies Property 1.

Algorithm for an iteration

- Construct *N'(I, J)*
- Raise prices in *J*
- New edge enters *N*
- OR a subset in I becomes tight

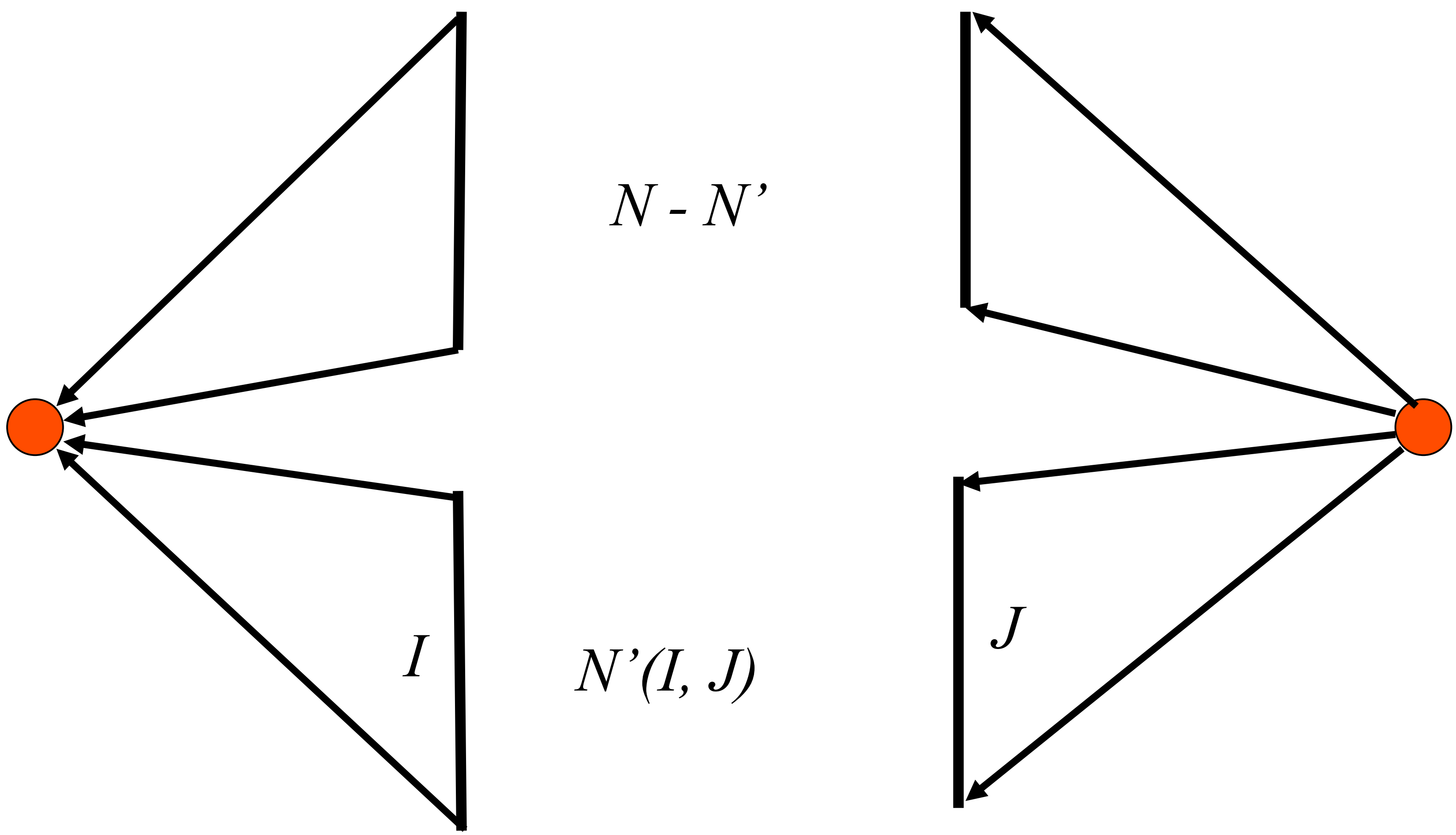
Network $N(p)$



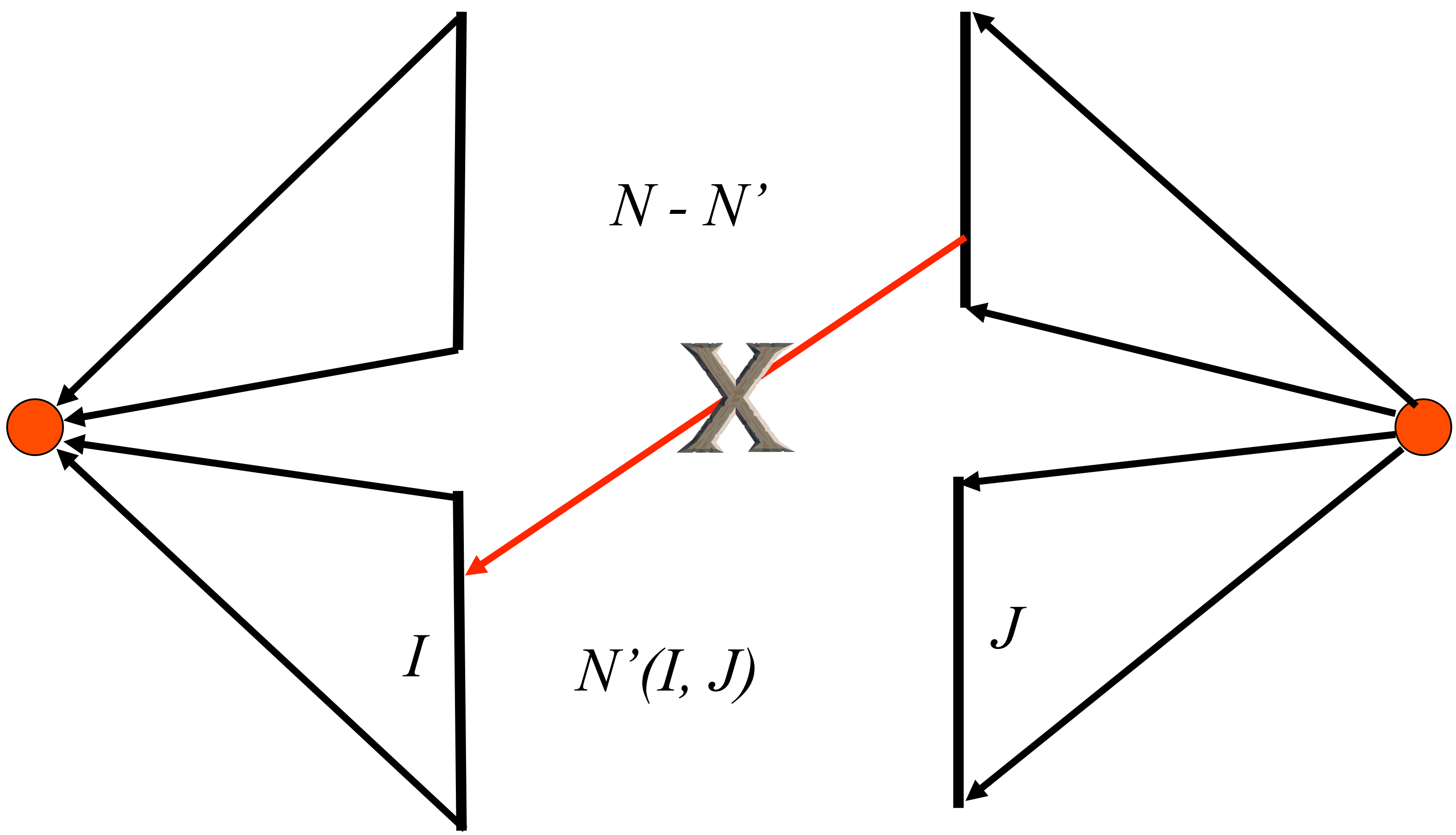
- Construct $N'(I, J)$
  - Find a balanced flow in $N(p)$

    Let d = max surplus w.r.t. balanced flow
  - I = buyers with surplus d
  - J = goods desired by I
- Raise prices in J
- New edge enters N
- OR a subset in I becomes tight

Network $N(\underline{p})$

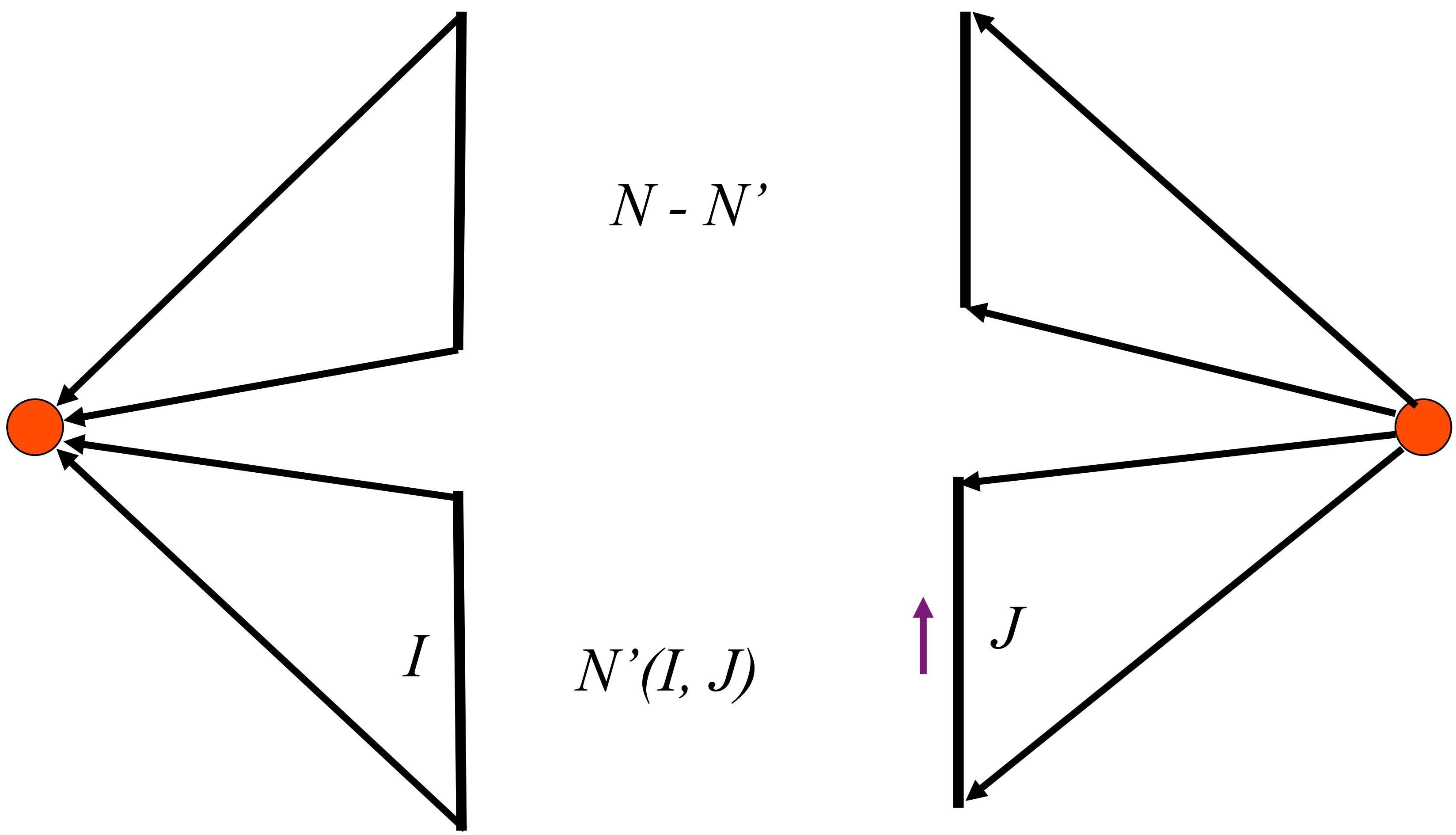


Network $N(\underline{p})$

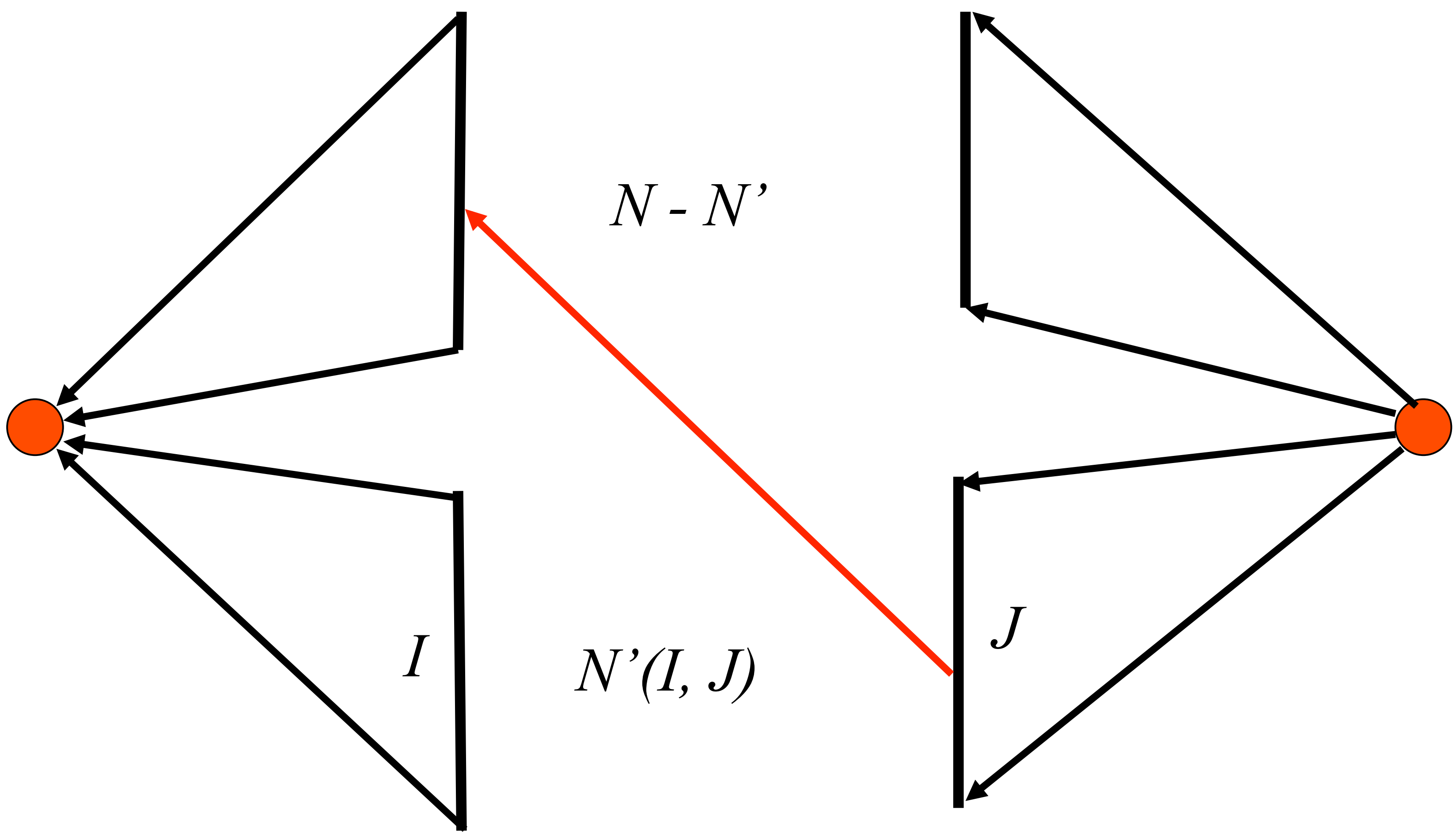


- Construct *N'(I, J)*
- Raise prices in *J*
    - *N'* is decoupled from *N - N'*
- New edge enters *N*
- OR a subset in I becomes tight

Network $N(\underline{p})$

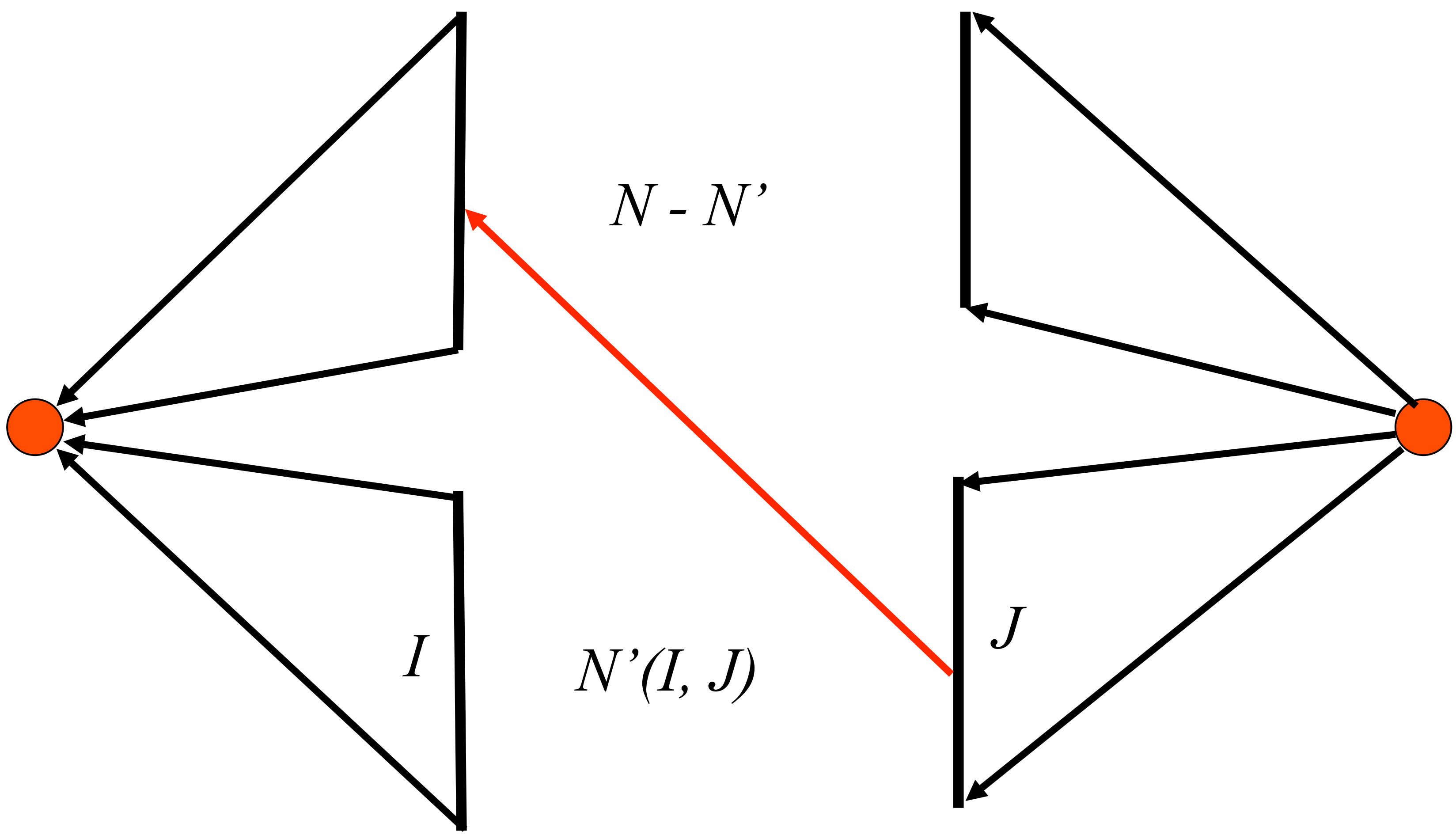


Network $N(\underline{p})$



By Property 1, this edge did not carry any flow.

Network *N(p)*



Hence Invariant is not violated by its removal.

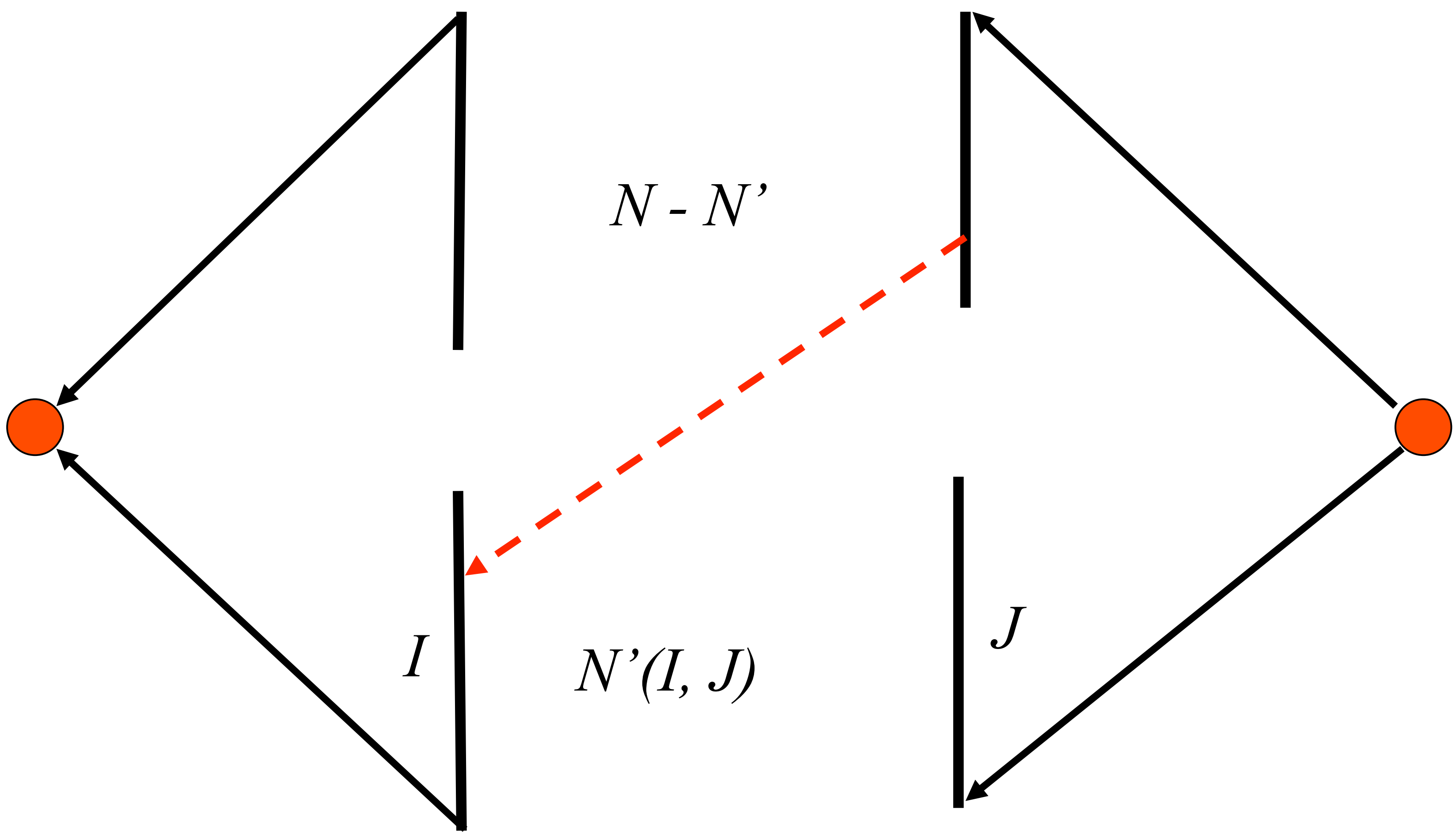
Raise prices in J

- proportionately, so that
  edges in N' don't change.

- $p \,.\, x$, for each p in J
  - initialize $x = 1$
  - raise x

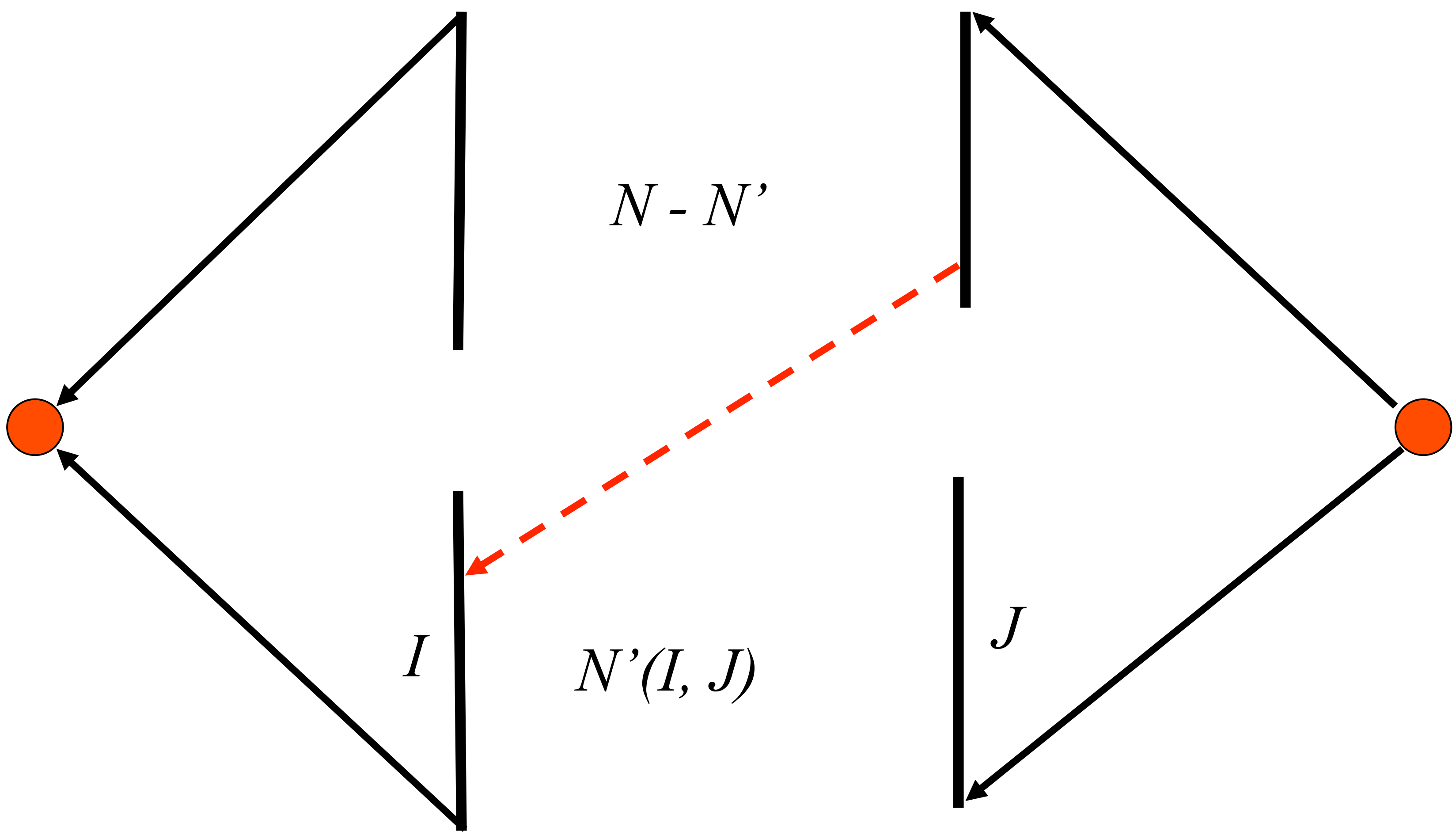
- Construct *N'(I, J)*
- Raise prices in *J*
- New edge enters *N*
- OR a subset in I becomes tight

Network $N(\underline{p})$

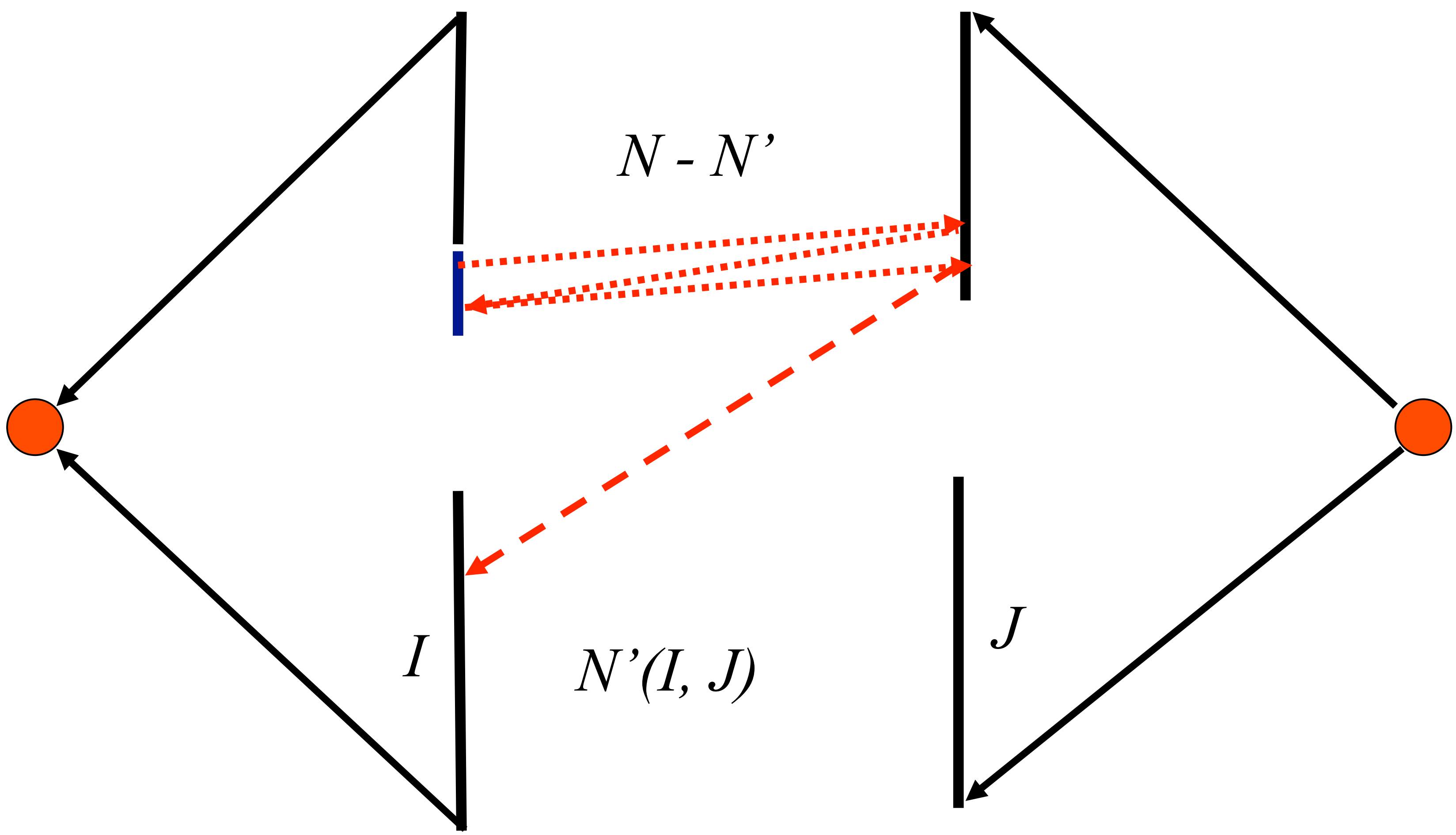


- Construct *N'(I, J)*
- Raise prices in *J*
- New edge enters *N*
  - Recompute balanced flow
  - Buyers in *N* - *N'* having residual paths to *N'* → Move to *N'*
- OR a subset in I becomes tight

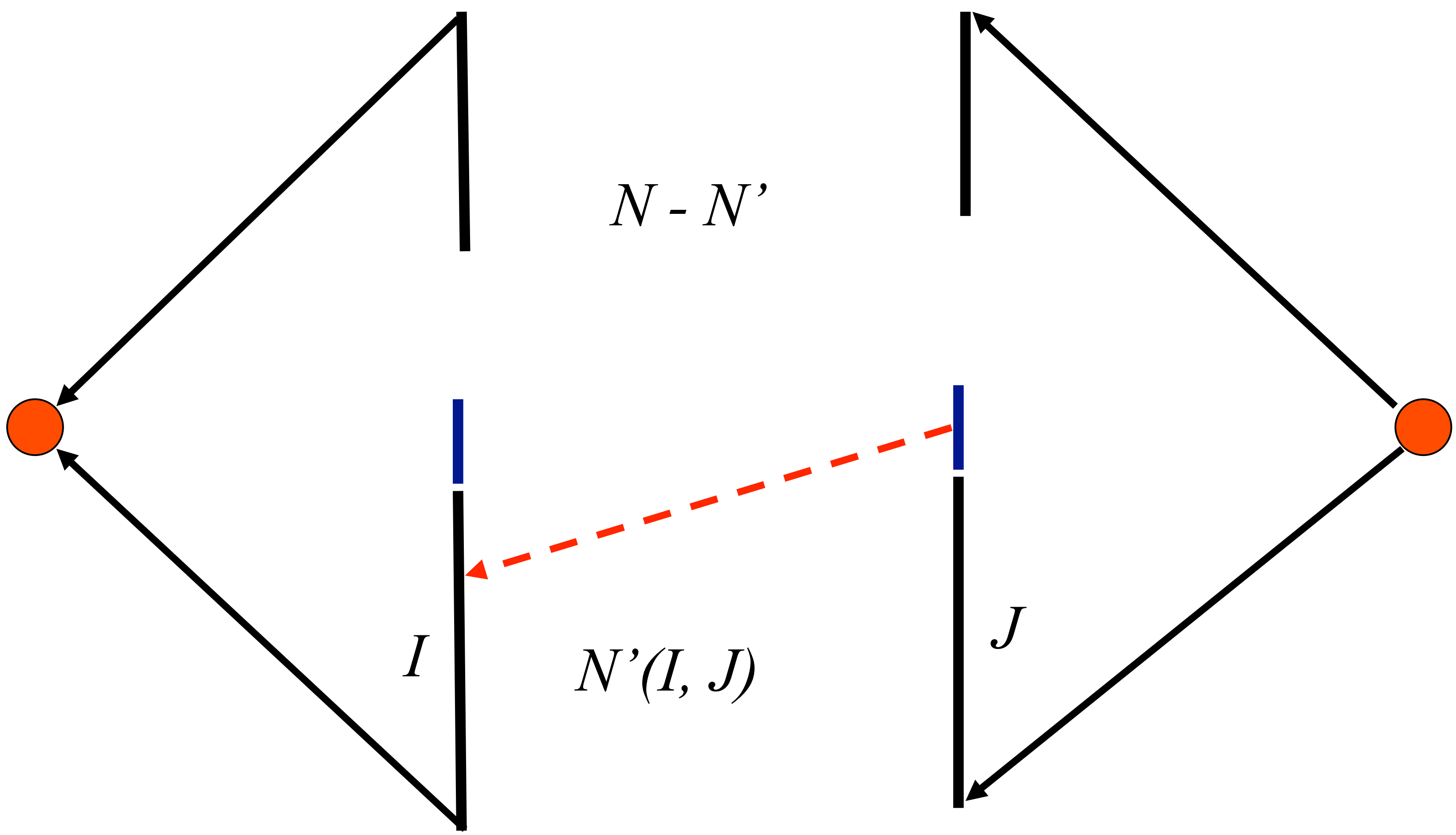
Network $N(\underline{p})$



Network $N(\underline{p})$



Network *N(p)*

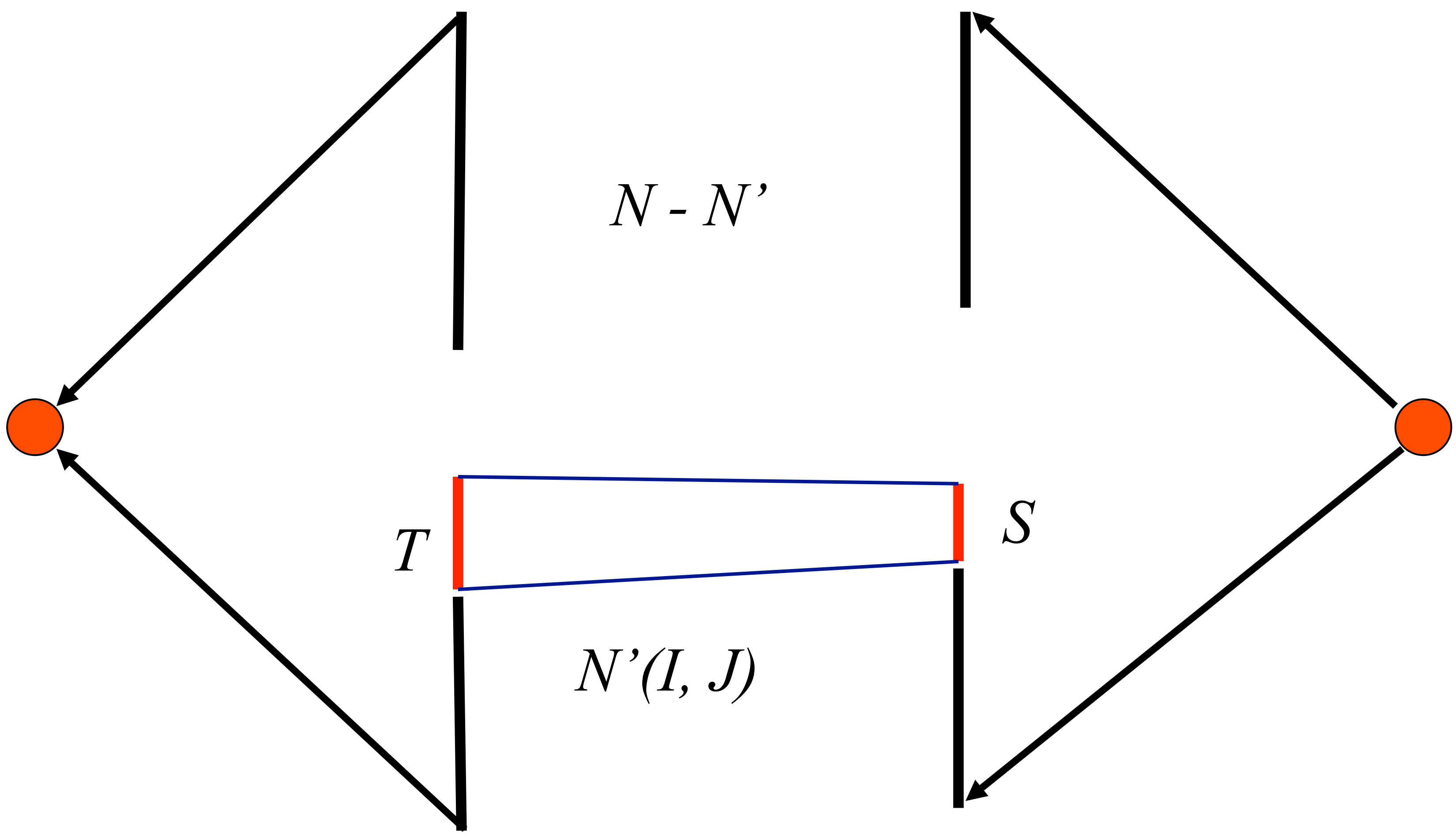


- Construct $N'(I, J)$
- Raise prices in J
- New edge enters N
  - Recompute balanced flow
  - Buyers moved to N' will have sufficiently large surplus
- OR a subset in I becomes tight

Algorithm for an iteration

- Construct $N'(I, J)$
- Raise prices in J
- New edge enters N
- OR a subset in I becomes tight

Tight set: $p(S) = m(T)$



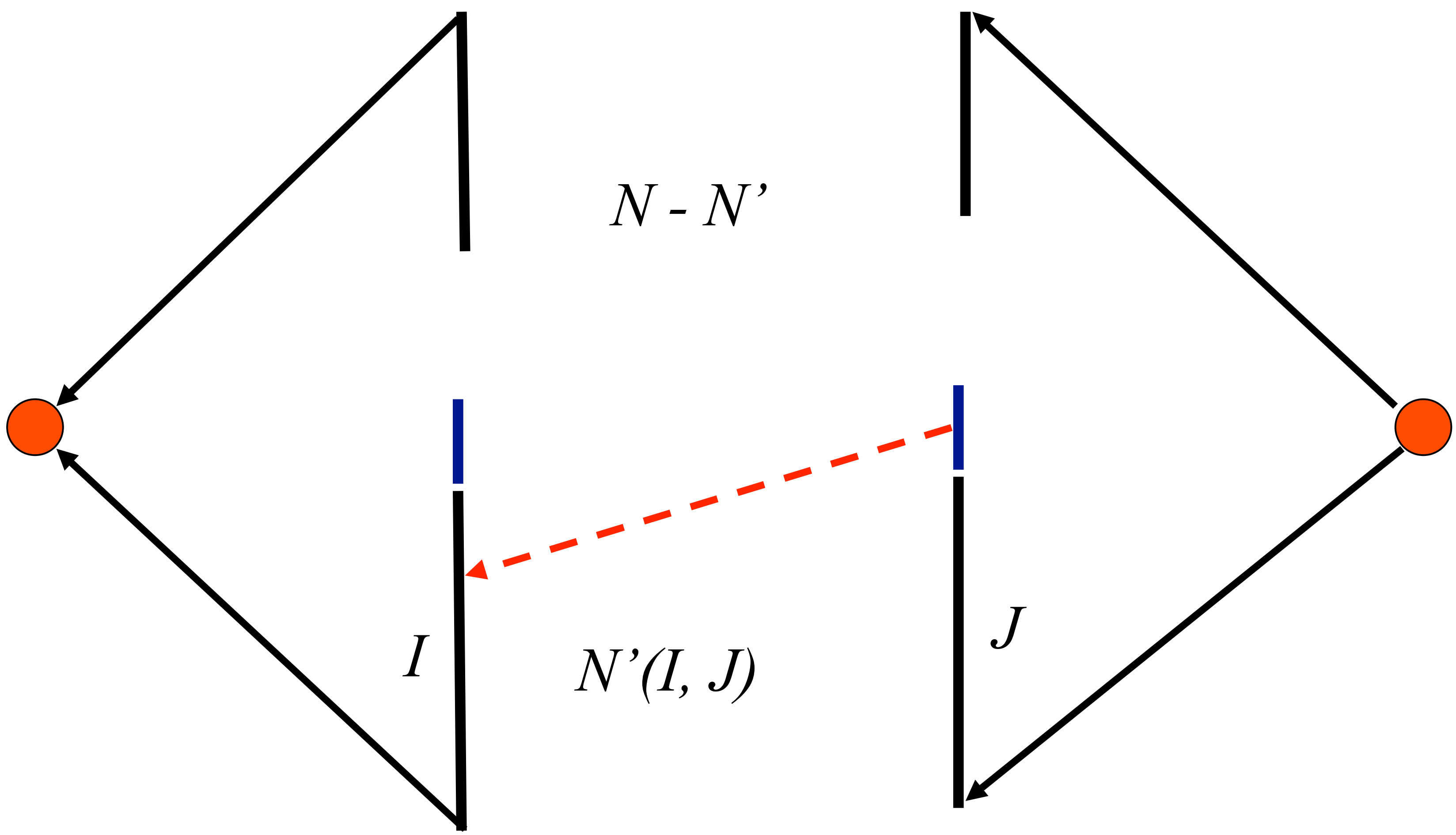
- Surplus of buyers in T drops to 0

- Assume k sub-iterations.

- Let $d_0 = d$. At the end of l^{th} sub-iteration,

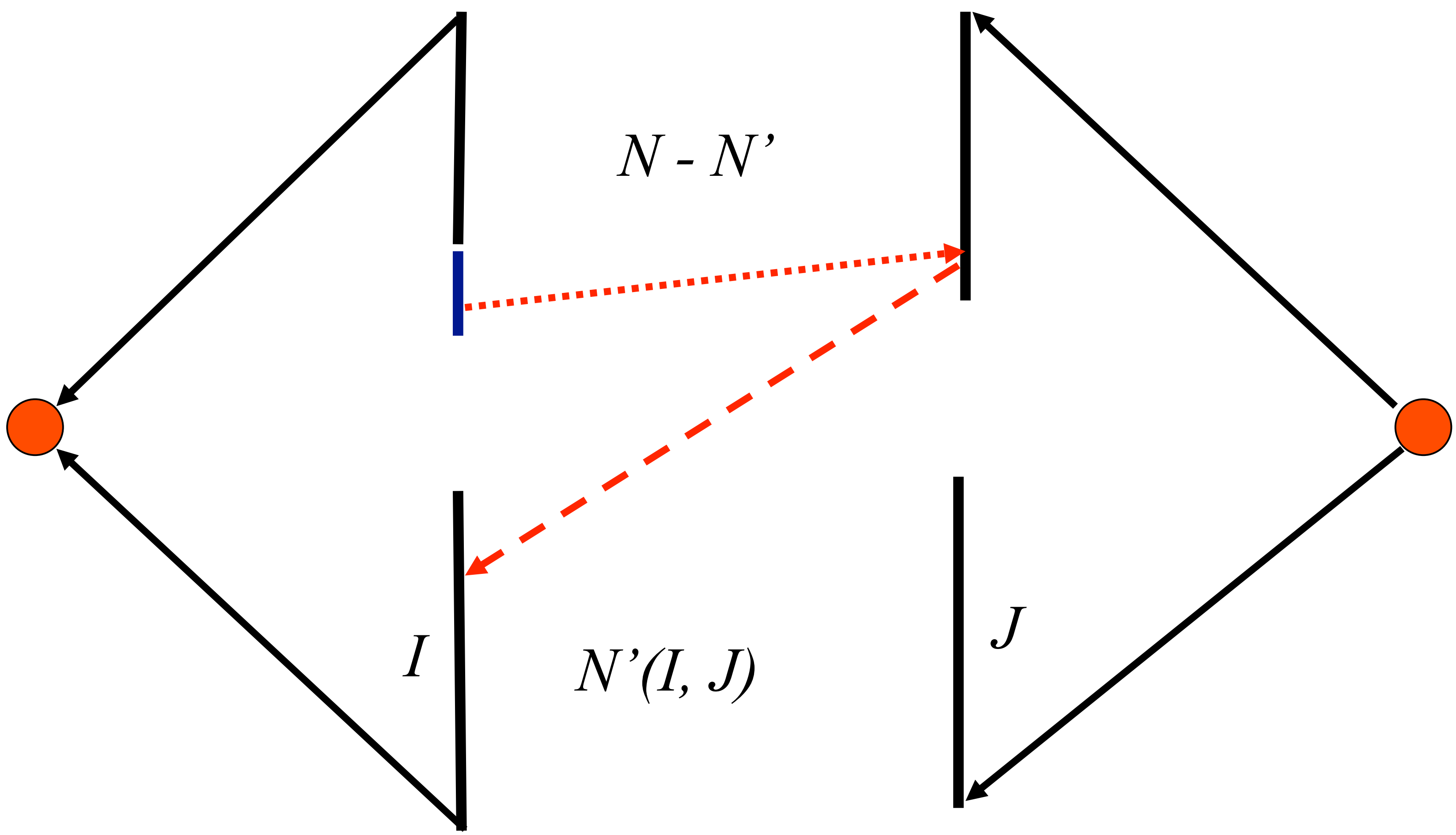
$d_l = \min\{\text{surplus}(i) \mid i \text{ is in } I\}$. So, $d_k = 0$.

Network $N(\underline{p})$



- Construct *N'(I, J)*
- Raise prices in *J*
- New edge enters *N*
  - Recompute balanced flow
  - Move buyers in *N - N'* having residual paths to *N'*
    - will have sufficiently large surplus
- OR a subset in I becomes tight

Network $\underline{N(p)}$



- Assume k sub-iterations.

- Let $d_0 = d$. At the end of l^{th} sub-iteration,

$d_l = \min \{\text{surplus}(i) \mid i \text{ is in } I\}$. So, $d_k = 0$.

- ~~Decrease in l_1 norm in sub-iteration l is at least $(d_{l-1} - d_l)$~~

- Decrease in l_2^2 norm in sub-iteration l is at least $(d_{l-1} - d_l)^2$

Our algorithm

- Reduces l_2 norm of surplus vector by $1/n^2$ fraction in each iteration

- Polynomial time algorithm

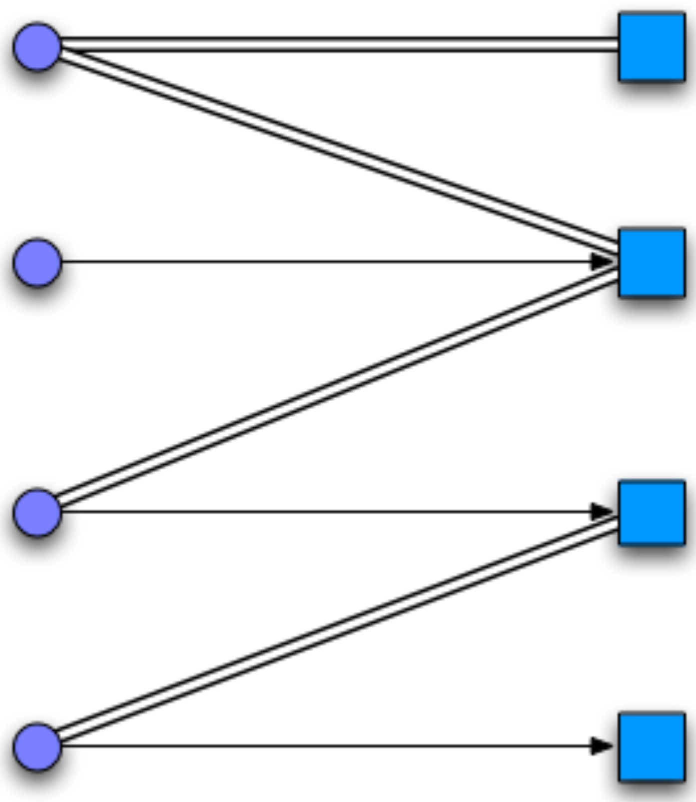
An improved algorithm by Orlin ‘10

- More intuitive
- Scaling algorithm
- Similar to Hungarian algorithm for matchings

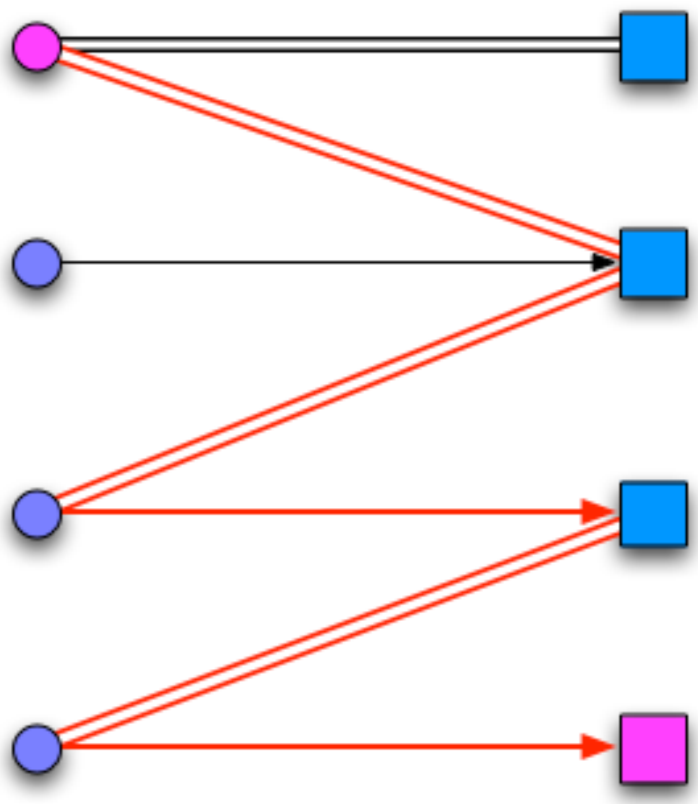
The residual network N(p)

- In the bipartite graph from buyer to goods
- For each equality edge (i,j), (i,j)$\in$N(p)
- For every (i,j) with $x_{ij}>0$, there is an arc (j,i)$\in$N(p)

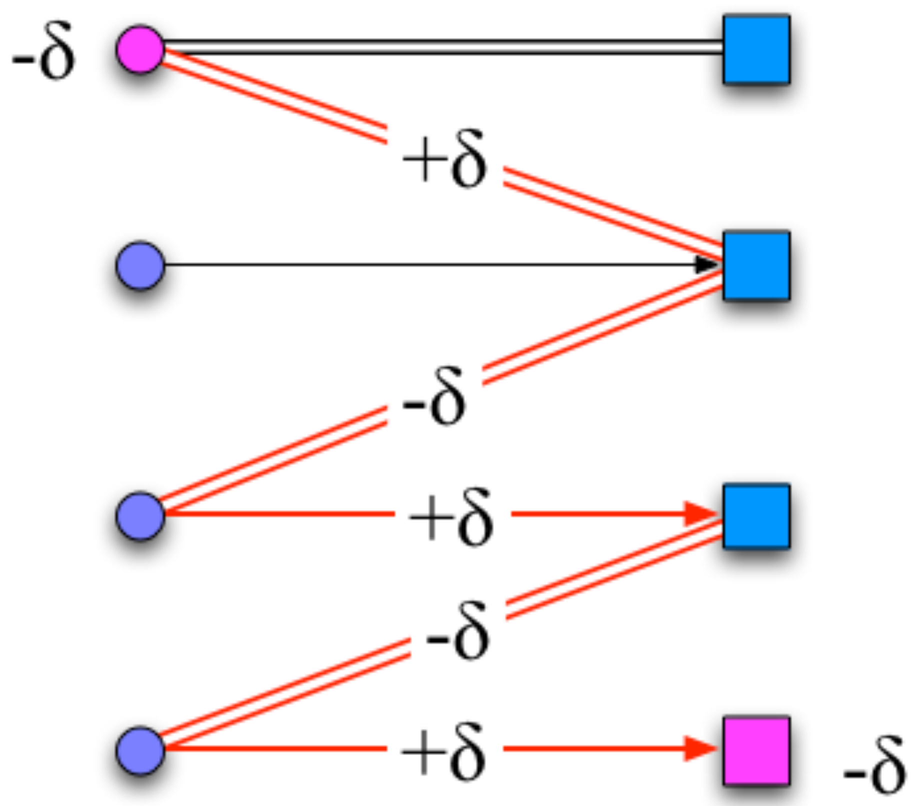
An example



For an augmenting path connecting a buyer and a goods both with surplus



For an augmenting path connecting a buyer and a goods both with surplus

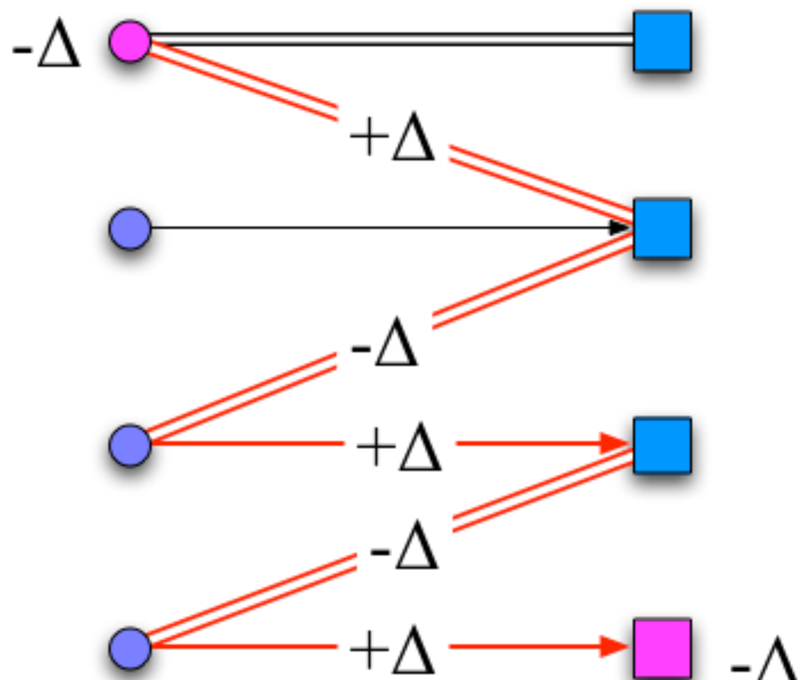


- In each scale, we maintain:
  - The surplus of every buyer ≥ 0
  - The surplus of every goods between $[0,\Delta]$
  - All x_{ij} are multiples of Δ

- Our aim in each scale:
  - The surplus of every buyer $<\Delta$

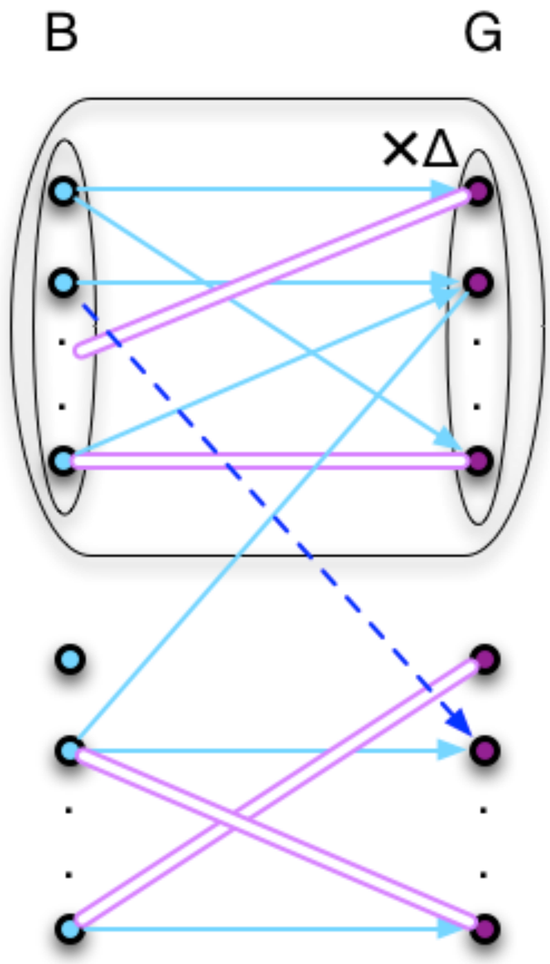
- After that reduce Δ by one half

- In each phase, start from buyers with surplus $\geq\Delta$
- Find augmenting paths to goods with surplus $\geq\Delta$



- If there is no such augmenting paths,
- Define the ActiveSet to be the vertices reachable from buyers with surplus $\geq \Delta$
  - Multiply the prices of goods in ActiveSet by the same factor q
  - Until ActiveSet becomes larger or some goods in ActiveSet has surplus $\geq \Delta$

- ActiveSet can become larger, since the goods not in it becomes more attractive



- It takes O(m+nlog n) to find an augmenting path from a buyer

- In each phase, the sum of all surplus is O(nΔ), so we just need to find O(n) augmenting paths
  - Because we start from the allocation of the previous phase with 2Δ

- Total running time: $O(n(m+n\log n)(m_{max}+n\log U_{max}))$
  - Δ begins with m_{max}/n, and ends with $1/(8n^2U_{max})$

Arrow-Debreu market

- Each agent is assigned a bundle of goods instead of an amount of money
- They sell their own goods to other agents then buy their desirable goods
- Fisher market can be seen as a special case of AD market
- Still no combinatorial polynomial algorithms!