Recall that a dominating set in a graph \( G = (V, E) \) is a set \( X \) of vertices such that for every \( v \in V \), either \( v \in X \) or \( v \) has a neighbour in \( X \). The DOMINATING SET problem is, given \( G \) and \( k \), to decide whether \( G \) has a dominating set of size at most \( k \).

**Exercise 1:** Problems on \( d \)-degenerate graphs \((5+5+5=15 \text{ points})\)

A graph is \( d \)-degenerate if every induced subgraph contains a vertex of degree at most \( d \). This is quite a general notion of sparseness, but still allows for some algorithmic conclusions.

(a) Show that the “degeneracy number”, i.e., the smallest \( d \) such that \( G \) is \( d \)-degenerate, can be computed in polynomial time.

(b) Show that INDEPENDENT SET and DOMINATING SET have constant-factor approximations on \( d \)-degenerate graphs.

(c) Show that for INDEPENDENT SET, we also get a linear kernel.

**Exercise 2:** Dominating Set by bounded treewidth \((10 \text{ points})\)

Show that DOMINATING SET can be solved in \( 2^{O(k)} n^{O(1)} \) time for graphs of treewidth \( k \).

**Exercise 3:** More treewidth \((5+10=15 \text{ points})\)

Show the following.

(a) Let \( u \) and \( v \) be two vertices of a graph \( G \), connected by a flow of more than \( k + 1 \) vertex-disjoint paths. Then any tree decomposition of \( G \) of width at most \( k \) must contain a bag \( X \) with \( u, v \in X \).

(b) Let \( G = (V, E) \) be a planar graph, \( v \in V \), and \( N_d[v] \) be the set of vertices at distance at most \( d \) of \( v \) (including \( v \) itself). Then \( G[N_d[v]] \) has treewidth bounded by \( O(d) \). (Hint: Go via outerplanarity.)