Exercise 1 (20 points)

The quality of the lower bound given by a linear programming relaxation is usually measured as the ratio between $Z_{IP}$, the optimal value of the original integer programming problem, and $Z_{LP}$, the optimal value of its LP relaxation. This ratio is known as the integrality gap of the given integer programming problem.

Usually one is interested in bounding the integrality gap of classes of IP problems. E.g., a long-standing open question is: What is the largest possible integrality gap that may arise in the subtour IP formulation (see next page) of any symmetric metric TSP instance? (‘Metric’ means that the edge weights satisfy the triangle inequality.) It is known that this value lies between 4/3 and 3/2. In particular, solving the LP relaxation of the subtour IP always gives a lower bound that is off by a factor of at most 3/2.

We will discuss the upper bound of 3/2 in the lecture; this exercise concerns the lower bound of 4/3. Consider the (unweighted) graph $G' = (V, E')$ in Figure 1, and define a metric TSP instance $G = (V, E), c : E \rightarrow \mathbb{N}_0$ as follows: $E := \binom{V}{2}$ (i.e. $G$ is complete), and for $u, v \in V, u \neq v$, the weight $c_e$ of the edge $e = \{u, v\}$ is given by the length (number of edges) of a shortest path from $u$ to $v$ in $G'$. 

a) Show that the weight function $c$ indeed satisfies the triangle inequality.

b) Show that the length of a shortest salesman tour in $G$ (i.e. $Z_{IP}$) is $(4 + o(1))\ell$ as $\ell \rightarrow \infty$.

*Hint:* Think of a tour in $G$ as a closed walk in $G'$ that visits all vertices.

c) Show that the value of the optimal solution of the linear programming relaxation of the subtour IP formulation (i.e. $Z_{LP}$) is $(3 + o(1))\ell$ as $\ell \rightarrow \infty$. 

*please turn over*
The subtour IP formulation of TSP is:

\[
\begin{align*}
& \text{maximize} \quad \sum_{e \in E} c_e x_e \\
& \text{s.t.} \quad \sum_{e \in \delta(v)} x_e = 2, \quad v \in V, \\
& \quad \sum_{e \in E(S)} x_e \leq |S| - 1, \quad \emptyset \neq S \subset V, \\
& \quad x_e \in \{0, 1\}, \quad e \in E,
\end{align*}
\]

where \(\delta(v)\) denotes the set of edges incident to \(v\), and \(E(S)\) denotes the set of edges with both endpoints in \(S\).

The LP relaxation is obtained by replacing the last constraint with

\[
0 \leq x_e \leq 1, \quad e \in E.
\]

**Exercise 2 (12 points)**

A proper vertex-coloring of a graph \(G\) is a coloring of the vertices of \(G\) in which no two adjacent vertices receive the same color.

a) For a given graph \(G\) and an integer \(k \geq 2\), give an integer program that is feasible if and only if \(G\) allows a proper vertex-coloring using at most \(k\) colors. The size of your program should be polynomial in the size of \(G\).

b) Consider now the LP relaxation of your integer program. Give an example of a graph \(G\) and an integer \(k\) for which the LP relaxation is feasible but the original integer program is not.