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Exercises for Optimization

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss12/OPT

Exercise sheet 2 Due: **Tuesday, May 8, 2012**

You need to collect at least 50% of all points on the first six exercise sheets, and at least 50% of all points on the remaining exercise sheets. You are allowed to hand in homework in teams of two.

Exercise 1 (*15 points*) Consider the standard form polyhedron

$$P = \{x \in \mathbb{R}^4 \mid Ax = b, x \geq 0\},$$

where

$$A := \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$

a) List all basic solutions of P. (Note that there are some degeneracies. You will find out that there is only one basic solution that is feasible – don't let this confuse you.)

Hint: Recall that the inverse of $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is

$$B^{-1} = \frac{1}{\det(B)} \cdot \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix},$$

where $\det(B) = b_{11}b_{22} - b_{12}b_{21}$ is the determinant of B. (This is a special case of Cramer's rule, see e.g. Wikipedia.)

b) In the lecture we have mentioned (but not proved) that if P is non-empty and bounded, then for any vector $c \in \mathbb{R}^4$ there is a basic feasible solution $x \in P$ (a "corner") that minimizes $c^T x$ among all points in P.

Show that P is non-empty and bounded.

c) What can you conclude from a) and b) about the geometric shape of P? Can you find a more direct argument for your observation?

Exercise 2 (*10 points*) Let $A \in \mathbb{R}^{m \times n}$ be a matrix with rows a_i^T, $i = 1, \dots, m$, and let $J \subseteq [n]$, $|J| = n - m$, be a subset of column indices. (In the lecture we also made the assumption that the rows of A are linearly independent, but this is irrelevant here.)

Let $C \in \mathbb{R}^{n \times n}$ be the matrix with rows a_i^T, $i = 1, \dots, m$, and e_j^T, $j \in J$, where $e_j \in \mathbb{R}^n$ denotes the j-th unit vector.

Prove that the n columns $C_1, \dots, C_n$ of C are linearly independent (i.e. C is regular) if and only if the m columns A_j, $j \notin J$, of A are linearly independent.

Exercise 3 (*12 points*) (adapted from BT, Exercise 2.10)

Consider an LP in standard form "minimize $c^T x$ s.t. $Ax = b$ and $x \geq 0$", where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and the m rows of A are linearly independent. For each of the following statements, decide whether it is true in general. If it is, provide a short proof, otherwise, give a counterexample.

a) The set of all optimal solutions is bounded.

b) At every optimal solution, no more than m variables are positive.

c) If there is more than one optimal solution, then there are infinitely many optimal solutions.

d) If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.

Exercise 4 (*5 points, extra credit*)

Consider the standard form polyhedron

$$P = \{x \in \mathbb{R}^3 \mid Ax = b, x \geq 0\},$$

where

$$A := \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Show that P has a basic feasible solution which is degenerate but "atypical" in the sense that it nevertheless corresponds to only one basis.