Reto Spöhel, Rob van Stee **Summer 2012**

Exercises for Optimization

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss12/OPT

Exercise sheet 3 Due: **Tuesday, May 15, 2012**

You need to collect at least 50% of all points on the first six exercise sheets, and at least 50% of all points on the remaining exercise sheets. You are allowed to hand in homework in teams of two.

Exercise 1 (*15 points*) Consider the linear program

$$\begin{aligned} \text{minimize} \quad & -5x_1 - 4x_2 - 3x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 8 \\ & 3x_1 + 4x_2 + 2x_3 \leq 11 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

a) Bring the problem into standard form by introducing slack variables x_4, x_5, x_6.

b) Solve the standard form problem using the simplex method. Start from the basic feasible solution $x_1 = x_2 = x_3 = 0$ (i.e., with the slack variables as basis) and, in each iteration, choose as the entering variable the variable with *lowest index* among all variables that correspond to improving basic directions.

   The problem contains no degeneracies. The optimal solution has a value of -13, and once you reach it, you are allowed to stop without explicitly checking its optimality. Make sure to check after each iteration that what you have is indeed a bfs with strictly lower value than the previous one.

Exercise 2 (*10 points*) (BT, Exercise 3.4)

Consider the (non-standard form) polyhedron

$$P = \{x \in \mathbb{R}^n \mid Ax = b, Dx \leq f, Ex \leq g\},$$

where A, D, E are matrices and b, f, g are vectors of appropriate dimensions. Let $x^* \in P$ be a vector with $Dx^* = f$ and $Ex^* < g$ (where as usual the latter is understood as a *component-wise* strict inequality). Show that the set of feasible directions at x^* is

$$\{d \in \mathbb{R}^n \mid Ad = 0, Dd \leq 0\}.$$

Exercise 3 (*15 points*) (adapted from BT, Exercises 3.3 & 3.7)

Let x^* be a point of the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. As usual, our goal is to minimize $c^T x$ over P, for some given vector $c \in \mathbb{R}^n$.

a) Show that a vector $d \in \mathbb{R}^n$ is a feasible direction at x^* if and only if $Ad = 0$ and $d_i \geq 0$ for every i such that $x_i^* = 0$.

b) Let $Z = \{i \mid x_i^* = 0\}$. Show that x^* is an optimal solution if and only if the LP

$$\begin{array}{lll} \text{minimize} & c^T d & \\ \text{s.t.} & Ad = 0 & \\ & d_i \geq 0, & i \in Z, \end{array}$$

has an optimal cost of zero.