You need to collect at least 50% of all points on the first six exercise sheets, and at least 50% of all points on the remaining exercise sheets. You are allowed to hand in homework in teams of two.

**Exercise 1** *(15 points)* Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad -5x_1 - 4x_2 - 3x_3 \\
\text{s.t.} & \quad 2x_1 + 3x_2 + x_3 \leq 5 \\
& \quad 4x_1 + x_2 + 2x_3 \leq 8 \\
& \quad 3x_1 + 4x_2 + 2x_3 \leq 11 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

a) Bring the problem into standard form by introducing slack variables \(x_4, x_5, x_6\).

b) Solve the standard form problem using the simplex method. Start from the basic feasible solution \(x_1 = x_2 = x_3 = 0\) (i.e., with the slack variables as basis) and, in each iteration, choose as the entering variable the variable with lowest index among all variables that correspond to improving basic directions.

The problem contains no degeneracies. The optimal solution has a value of \(-13\), and once you reach it, you are allowed to stop without explicitly checking its optimality. Make sure to check after each iteration that what you have is indeed a bfs with strictly lower value than the previous one.
Exercise 2 (10 points) (BT, Exercise 3.4)

Consider the (non-standard form) polyhedron

\[ P = \{ x \in \mathbb{R}^n \mid Ax = b, Dx \leq f, Ex \leq g \}, \]

where \( A, D, E \) are matrices and \( b, f, g \) are vectors of appropriate dimensions. Let \( x^* \in P \) be a vector with \( Dx^* = f \) and \( Ex^* < g \) (where as usual the latter is understood as a component-wise strict inequality). Show that the set of feasible directions at \( x^* \) is

\[ \{ d \in \mathbb{R}^n \mid Ad = 0, Dd \leq 0 \}. \]

Exercise 3 (15 points) (adapted from BT, Exercises 3.3 & 3.7)

Let \( x^* \) be a point of the standard form polyhedron \( P = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \} \). As usual, our goal is to minimize \( c^T x \) over \( P \), for some given vector \( c \in \mathbb{R}^n \).

a) Show that a vector \( d \in \mathbb{R}^n \) is a feasible direction at \( x^* \) if and only if \( Ad = 0 \) and \( d_i \geq 0 \) for every \( i \) such that \( x_i^* = 0 \).

b) Let \( Z = \{ i \mid x_i^* = 0 \} \). Show that \( x^* \) is an optimal solution if and only if the LP

\[
\begin{align*}
\text{minimize} & \quad c^T d \\
\text{s.t.} & \quad Ad = 0 \\
& \quad d_i \geq 0, \quad i \in Z,
\end{align*}
\]

has an optimal cost of zero.