Exercises for Optimization

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss12/OPT

Exercise sheet 4

Due: Tuesday, May 22, 2012

You need to collect at least 50% of all points on the first six exercise sheets, and at least 50% of all points on the remaining exercise sheets. You are allowed to hand in homework in teams of two.

Exercise 1 (20 points) Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad -5x_1 - 4x_2 - 3x_3 \\
\text{s.t.} & \quad 2x_1 + 3x_2 + x_3 \leq 5 \\
& \quad 4x_1 + x_2 + 2x_3 \leq 8 \\
& \quad 3x_1 + 4x_2 + 2x_3 \leq 11 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

On the last exercise sheet, you have brought this into standard form and solved it using the simplex method.

\begin{itemize}
\item[a)] Redo this exercise using the revised simplex method.
\item[b)] Redo this exercise using the full tableau implementation.
\end{itemize}

As last week, start from the basic feasible solution \(x_1 = x_2 = x_3 = 0\) (i.e., with the slack variables as basis) and use Bland’s rule for pivot selection. The optimal solution is of course still \(-13\), but this time we also want you to check its optimality (in the revised simplex, resp. the full tableau framework).

Note that you have many sanity checks available, in particular if you kept a copy of last week’s solution. Use these!

(For a fairly detailed numerical example of the full tableau implementation, see Example 3.5 in the book.)
Exercise 2 (20 points)

In this exercise, we go into some more detail about the problem of finding an initial bfs for the Simplex Method. Consider the standard form LP “minimize $c^T x$ s.t. $Ax = b$ and $x \geq 0$”, where

$$A := \begin{bmatrix} 0 & 2 & 1 & 1 \\ -4 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 0 \\ 3/2 \end{bmatrix}, \quad \text{and} \quad c^T = [2 \ 1 \ 3 \ 1].$$

Note that it’s not entirely trivial to come up with an initial bfs, even though you still might find one ‘by hand’ if you look hard.

a) Formulate the auxiliary LP as discussed in the lecture (introducing artificial variables $y_1, y_2, y_3$).

b) Solve the auxiliary LP with the full tableau implementation of the simplex method. This should take two iterations, and the second iteration will be degenerate. 

Make sure you start with the correct initial tableau! Note that the vector $c^T$ given above is irrelevant for the time being.

c) If you did the previous part correctly, you should now have $x_2, x_3, y_3$ as basic variables, and of course all reduced costs are nonnegative since the simplex method terminated. Moreover, $y_3$ is zero; i.e., the current bfs of the auxiliary problem is degenerate. The values of $x_2$ and $x_3$ give a feasible solution to the original problem (even a basic one, as we will see), but we do not yet have an associated basis as $y_3$ is not a variable of the original problem.

To drive $y_3$ out of the basis, we need to do one more basis change. This will again be a degenerate basis change, i.e., one that does not change the current bfs but only the associated basis. Because of this we can ignore the reduced costs for the moment – as we will see, we do not really “move” anyway, they do not matter.

(i) Try performing a basis change with $x_1$ as the entering and $y_3$ as the leaving variable. Why does this not work?

(ii) Try performing a basis change with $x_4$ as the entering and $y_3$ as the leaving variable. Why does this work? Give the resulting tableau.

d) Consider now the general setting, and assume that none of the original variables that are not basic yet can be brought into the basis, because they all behave like $x_1$ in the above. What does this imply for the matrix $A$?

e) After part c), you now have a final tableau with $x_2, x_3, x_4$ as the basic variables.

(i) If you did everything correctly, the zero-th row should have a very special structure. Argue why it must have this structure.

(ii) What is encoded in the last three columns of the tableau (the ones we are going to drop in a moment)?

f) We now can drop the artificial variables and start solving the original problem with the initial bfs we found. (The feasible solution we found is a bfs because we explicitly computed a basis for it!) Give the tableau for the original problem we start solving now. (Note that this involves bringing back in the original objective function encoded by $c^T$.)