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SAARLAND UNIVERSITY
COMPUTER SCIENCE

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Exercises for Optimization

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss12/OPT

Exercise sheet 4 Due: **Tuesday, May 22, 2012**

You need to collect at least 50% of all points on the first six exercise sheets, and at least 50% of all points on the remaining exercise sheets. You are allowed to hand in homework in teams of two.

Exercise 1 (*20 points*) Consider the linear program

$$\begin{aligned} \text{minimize} \quad & -5x_1 - 4x_2 - 3x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 8 \\ & 3x_1 + 4x_2 + 2x_3 \leq 11 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

On the last exercise sheet, you have brought this into standard form and solved it using the simplex method.

a) Redo this exercise using the revised simplex method.

b) Redo this exercise using the full tableau implementation.

As last week, start from the basic feasible solution $x_1 = x_2 = x_3 = 0$ (i.e., with the slack variables as basis) and use Bland's rule for pivot selection. The optimal solution is of course still -13, but this time we also want you to check its optimality (in the revised simplex, resp. the full tableau framework).

Note that you have many sanity checks available, in particular if you kept a copy of last week's solution. Use these!

(For a fairly detailed numerical example of the full tableau implementation, see Example 3.5 in the book.)

Exercise 2 (*20 points*)

In this exercise, we go into some more detail about the problem of finding an initial bfs for the Simplex Method. Consider the standard form LP "minimize $c^T x$ s.t. $Ax = b$ and $x \geq 0$", where

$$A := \begin{bmatrix} 0 & 2 & 1 & 1 \\ -4 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 0 \\ 3/2 \end{bmatrix}, \qquad \text{and} \qquad c^T = \begin{bmatrix} 2 & 1 & 3 & 1 \end{bmatrix}.$$

Note that it's not entirely trivial to come up with an initial bfs, even though you still might find one 'by hand' if you look hard.

a) Formulate the auxiliary LP as discussed in the lecture (introducing artificial variables y_1, y_2, y_3).

b) Solve the auxiliary LP with the full tableau implementation of the simplex method. This should take two iterations, and the second iteration will be degenerate.

   Make sure you start with the correct initial tableau! Note that the vector c^T given above is irrelevant for the time being.

c) If you did the previous part correctly, you should now have x_2, x_3, and y_3 as basic variables, and of course all reduced costs are nonnegative since the simplex method terminated. Moreover, y_3 is zero; i.e., the current bfs of the auxiliary problem is degenerate. The values of x_2 and x_3 give a feasible solution to the original problem (even a basic one, as we will see), but we do not yet have an associated basis as y_3 is not a variable of the original problem.

   To drive y_3 out of the basis, we need to do one more basis change. This will again be a degenerate basis change, i.e., one that does not change the current bfs but only the associated basis. Because of this we can ignore the reduced costs for the moment – as we will not really "move" anyway, they do not matter.

   (i) Try performing a basis change with x_1 as the entering and y_3 as the leaving variable. Why does this *not* work?

   (ii) Try performing a basis change with x_4 as the entering and y_3 as the leaving variable. Why does this work? Give the resulting tableau.

d) Consider now the general setting, and assume that *none* of the original variables that are not basic yet can be brought into the basis, because they *all* behave like x_1 in the above. What does this imply for the matrix A?

e) After part c), you now have a final tableau with x_2, x_3, x_4 as the basic variables.

   (i) If you did everything correctly, the zero-th row should have a very special structure. Argue why it *must* have this structure.

   (ii) What is encoded in the last three columns of the tableau (the ones we are going to drop in a moment)?

f) We now can drop the artificial variables and start solving the original problem with the initial bfs we found. (The feasible solution we found is a bfs because we explicitly computed a basis for it!) Give the tableau for the original problem we start solving now. (Note that this involves bringing back in the original objective function encoded by c^T.)