Exercise 1 (10 points) Construct the dual of

\[
\begin{align*}
\min & \quad x_1 - x_3 \\
\text{s.t.} & \quad x_1 + 5x_2 - x_3 \leq 0 \\
& \quad 2x_1 - 2x_2 + 3x_2 = -1 \\
& \quad 3x_1 + x_2 - x_3 \geq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_3 \leq 0
\end{align*}
\]

Exercise 2 (10 points (adapted from BT 4.2))

a) Consider the following primal problem.

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \\
& \quad x \geq 0.
\end{align*}
\]

Give conditions under which this problem is equivalent to its own dual.

b) Construct a linear program with two variables that is infeasible and also has an infeasible dual. The coefficient matrix $A$ should have \textit{linearly independent} rows, and $|A_{ij}| \in \{1, 2\} \forall i, j$. (That is, this situation does not only occur if $A$ is not of full rank.) Your example does not have to be in standard form.
Exercise 3 (10 points (BT 4.3))

Suppose that we are given a subroutine which, given a system of linear inequalities, either produces a solution or decides that no solution exists.

Construct a simple algorithm that uses a single call to this subroutine and which finds an optimal solution to any linear programming problem that has an optimal solution.

Conclusion: linear programming is not harder than solving a system of linear inequalities (and therefore just as hard).

Hint: what was the main topic of the last two lectures?

Exercise 4 (10 points (BT 4.26))

Let $A$ be a given matrix. Show that exactly one of the following alternatives must hold.

a) There exists some $x \neq 0$ such that $Ax = 0$, $x \geq 0$.

b) There exists some $y$ such that $A^T y > 0$.

Note: for vectors, saying that it is $> 0$ means that every component is positive.

Exercise 5 (10 points (BT 4.28, extra credit))

Let $a$ and $a_1, \ldots, a_m$ be given vectors in $\mathbb{R}^n$. Prove that the following two statements are equivalent.

a) For all $x \geq 0$, we have $a^T x \leq \max_i a_i^T x$.

b) There exist nonnegative coefficients $\lambda_i$ that sum to 1 such that $a \leq \sum_{i=1}^m \lambda_i a_i$. 