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Exercises for Optimization

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss12/OPT

Exercise sheet 6 Due: **Tuesday, June 5, 2012**

You need to collect at least 50% of all points on the first six exercise sheets, and at least 50% of all points on the remaining exercise sheets. You are allowed to hand in homework in teams of two.

Exercise 1 (*10 points*) Construct the dual of

$$\begin{array}{lrcrcrcr}
\text{min} & x_1 & - & & & x_3 & & \\
\text{s.t.} & x_1 & + & 5x_2 & - & x_3 & \leq & 0 \\
 & 2x_1 & - & 2x_2 & + & 3x_2 & = & -1 \\
 & 3x_1 & + & x_2 & - & x_3 & \geq & 1 \\
 & x_1 & & & & & \geq & 0 \\
 & & & & & x_3 & \leq & 0
\end{array}$$

Exercise 2 (*10 points (adapted from BT 4.2)*)

a) Consider the following primal problem.

$$\begin{array}{rrcl}
\text{minimize} & c^T x & & \\
\text{subject to} & Ax & \geq & b \\
 & x & \geq & 0.
\end{array}$$

Give conditions under which this problem is equivalent to its own dual.

b) Construct a linear program with two variables that is infeasible and also has an infeasible dual. The coefficient matrix A should have *linearly independent* rows, and $|A_{ij}| \in \{1, 2\} \forall i, j$. (That is, this situation does not only occur if A is not of full rank.) Your example does not have to be in standard form.

Exercise 3 (*10 points (BT 4.3)*)

Suppose that we are given a subroutine which, given a system of linear inequalities, either produces a solution or decides that no solution exists.

Construct a simple algorithm that uses a **single** call to this subroutine and which finds an optimal solution to any linear programming problem that has an optimal solution.

Conclusion: linear programming is not harder than solving a system of linear inequalities (and therefore just as hard).

Hint: what was the main topic of the last two lectures?

Exercise 4 (*10 points (BT 4.26)*)

Let A be a given matrix. Show that exactly one of the following alternatives must hold.

a) There exists some $x \neq 0$ such that $Ax = 0, x \geq 0$.

b) There exists some y such that $A^T y > 0$.

Note: for vectors, saying that it is > 0 means that **every** component is positive.

Exercise 5 (*10 points (BT 4.28, extra credit)*)

Let a and $a_1, \ldots, a_m$ be given vectors in $\mathbb{R}^n$. Prove that the following two statements are equivalent.

a) For all $x \geq 0$, we have $a^T x \leq \max_i a_i^T x$.

b) There exist nonnegative coefficients λ_i that sum to 1 such that $a \leq \sum_{i=1}^m \lambda_i a_i$.