Exercise 1 (10 points (BT 7.3)) Each of \( n \) teams plays against every other team a total of \( k \) games. Assume that every game ends in a win or a loss (no draws) and let \( x_i \) be the number of wins of team \( i \). Let \( X \) be the set of all possible outcome vectors \((x_1, \ldots, x_n)\). Given an arbitrary vector \((x_1, \ldots, x_n)\), we would like to determine whether it belongs to \( X \), that is, whether it is a possible tournament outcome vector. Provide a network flow formulation of this problem.

Hint: if team \( n \) wins \( x_i \) times, it loses \((n - 1)k - x_i \) times.

Exercise 2 (10 points (BT 7.5))

Consider an uncapacitated network flow problem and assume that \( c_{ij} \geq 0 \) for all arcs. Let \( S_+ \) and \( S_- \) be the sets of source and sink nodes, respectively. Let \( d_{ij} \) be the length of the shortest directed path from node \( i \in S_+ \) to node \( j \in S_- \). Let \( d_{ij} = \infty \) if no path exists. We construct a transportation problem with the same source and sink nodes and the same values for the supplies and the demands. For every source node \( i \) and every sink node \( j \), we introduce a direct link with cost \( d_{ij} \). Show that the two problems have the same optimal cost.

Exercise 3 (10 points (BT 7.7))

Consider a network flow problem in which we impose an additional constraint \( f_{ij} \geq \ell_{ij} \) for every arc \((i, j)\). Construct an equivalent network flow problem in which there are no nonzero lower bounds on the arc costs.

Hint: let \( \bar{f}_{ij} = f_{ij} - \ell_{ij} \) and construct a new network for the arc flows \( \bar{f}_{ij} \). How should \( b_i \) be changed?
Exercise 4 (10 points (adapted from BT 7.8)) Consider a transportation problem in which all cost coefficients $c_{ij}$ are positive. Suppose that we increase the supply at some source nodes and the demand at some sink nodes (in a consistent way). Is it true that the value of the optimal cost will also increase? Prove or give a counterexample. What happens if some $c_{ij}$ values may be negative?