**Topic:** Fundamental concepts and algorithmic methods for solving linear and integer linear programs:
- Simplex method
- LP duality
- Ellipsoid method
- ...

**Prerequisites:** Basic math and theory courses

**Credit:** 4+2 hours => 9 CP; written end of term exam

**Lecturers:** Dr. Reto Spöhel, PD Dr. Rob van Stee

**Tutors:** Karl Bringmann, Ruben Becker

**Lectures:** Tue 10 –12 & Thu 12 –14 in E1.4 (MPI-INFINITY), room 0.24
**Exercises:** Wed 10 –12 / 14 –16
Optimization

Given:

- a **target function** \( f(x_1, ..., x_n) \) of decision variables \( x_1, ..., x_n \in \mathbb{R} \)
  
  - e.g. \( f(x_1, ..., x_n) = x_1 + 5x_3 - 2x_7 \)
  
  - or \( f(x_1, ..., x_n) = x_1 \cdot (x_2 + x_3) / x_8 \)

- a **set of constraints** the variables \( x_1, ..., x_n \) need to satisfy
  
  - e.g. \( x_1 \geq 0 \)

  and \( x_2 \cdot x_3 \leq x_4 \)

  and \( x_4 + x_5 = x_6 \)

  and \( x_1, x_3, x_5 \in \mathbb{Z} \)

Goal:

- find \( x_1, ..., x_n \in \mathbb{R} \) **minimizing** \( f \) subject to the given constraints

Geometric viewpoint: The set of points \( (x_1, ..., x_n) \in \mathbb{R}^n \) that satisfy all constraints is called the **feasible region** of the optimization problem.

I.e., find \( (x_1, ..., x_n) \) **inside the feasible region** that minimizes \( f \) among all such points
Optimization

• Important special cases:
  – **convex optimization**: Both the feasible region and the target function are **convex**
    ➔ any local minimum is a global minimum (!)
  – **Semi-Definite Programming**: constraints are **quadratic** and semidefinite; target function is **linear**
  – **Quadratic Programming**: constraints are **linear**, target function is **quadratic** and semidefinite
  – **Linear Programming**: constraints are **linear**, target function is **linear**
  – **Integer [Linear] Programming**: Linear programming with additional constraint that some or all decision variable must be **integral** (or even \( \in \{0,1\} \))

This course is mostly about LP and IP!
Linear Programming – A First Example

[Example taken from lecture notes by Carl W. Lee, U. of Kentucky]

- A company manufactures **gadgets** and **gewgaws**
- One kg of **gadgets**
  - requires **1 hour** of work, **1 unit** of wood, and **2 units** of metal
  - yields a **net profit of $5**
- One kg of **gewgaws**
  - requires **2 hours** of work, **1 unit** of wood, and **1 unit** of metal
  - yields a **net profit of $4**
- The company has **120 hours** of work, **70 units** of wood, and **100 units** of metal available
- What should it produce from these resources to **maximize its profit**?
Linear Programming – A First Example

- One kg of gadgets
  - requires 1 hour of work, 1 unit of wood, and 2 units of metal
  - yields a net profit of $5
- One kg of gewgaws
  - requires 2 hours of work, 1 unit of wood, and 1 unit of metal
  - yields a net profit of $4
- The company has 120 hours of work, 70 units of wood, and 100 units of metal available

\[ \begin{align*}
  \text{x}_1 & := \text{amount of gadgets} \\
  \text{x}_2 & := \text{amount of gewgaws} \\
  \text{max} \ z &= 5\text{x}_1 + 4\text{x}_2 \\
  \text{s.t.} \quad \text{x}_1 + 2\text{x}_2 & \leq 120 \\
  \text{x}_1 + \text{x}_2 & \leq 70 \\
  2\text{x}_1 + \text{x}_2 & \leq 100 \\
  \text{x}_1, \text{x}_2 & \geq 0
\end{align*} \]

- maximize profit
- work hours constraint
- wood constraint
- metal constraint
Linear Programming – A First Example

\[ \text{max } z = 5x_1 + 4x_2 \]
\[ \text{s.t. } x_1 + 2x_2 \leq 120 \]
\[ x_1 + x_2 \leq 70 \]
\[ 2x_1 + x_2 \leq 100 \]
\[ x_1, x_2 \geq 0 \]

- maximize profit
- work hours constraint
- wood constraint
- metal constraint

- all points with profit \(5x_1 + 4x_2 = 100\)
- all points with profit \(5x_1 + 4x_2 = 200\)
- all points with profit \(5x_1 + 4x_2 = 310\)

the only point with profit 310 inside the feasible region --- the optimum!

- \(x_2 = \text{amount of gewgaws}\)
- \(x_1 = \text{amount of gadgets}\)

- \(x_1 = 20 = 30\)
- \(x_1 = 50 = 70\)
- \(x_1 = 120\)
Observations / Intuitions

• It seems that in Linear Programming
  – the **feasible region** is always a **convex polygon/polytope**
  – the **optimum** is always attained at one of the **corners** of this polygon/polytope

• These intuitions are more or less true!
• In the coming weeks, we will **make them precise**, and exploit them to derive **algorithms for solving linear programs**.
  – First and foremost: **The simplex method**
Many other CS problems can be cast as LPs.

**Example:** MAXFLOW can be written as the following LP:

```
maximize \sum_{v:(s,v) \in E} f(s,v) \\
subject to \quad 0 \leq f(u,v) \leq c(u,v) \quad for all (u,v) \in E \\
and \quad \sum_{u:(u,v) \in E} f(u,v) = \sum_{w:(v,w) \in E} f(v,w) \quad for all v \in V \setminus \{s,t\}
```

- This LP has $2|E| + |V|-2$ many constraints and $|E|$ many decision variables.

LPs can be solved in time **polynomial** in the input size!

But the algorithm most commonly used for solving them in practice (simplex algorithm) is **not a polynomial algorithm**!
Books

• Main reference for the course:

*Introduction to Linear Optimization*
by Dimitris Bertsimas and John N. Tsitsiklis
- rather expensive to buy 😞
- the library has ca. 20 copies
- PDF of Chapters 1-5: google „leen stougie linear programming“

• Secondary reference:

*Combinatorial Optimization: Algorithms and Complexity*
by Christos H. Papadimitriou and Kenneth Steiglitz
- very inexpensive (ca. €15 on amazon.de) 😊
- old (1982) but covers most of the basics
  - classic!
Exercise sessions start **in week 3** of the semester

Two groups:
- Wed 10-12 in E1.7 (cluster building), room 001
  Tutor: Ruben
- Wed 14-16 in E 1.4 (this building), room 023
  Tutor: Karl

Course registration and assignment to groups next week!

Exercise sheets
- will be put online Tuesday evening each week
- should be handed in the following Tuesday in the lecture
- You need to achieve 50% of all available points in the first **and** in the second half of the term to be admitted to the exam.
That’s it for today.

On Thursday we will start with the definitions and theorems.