Polynomial-time approximation schemes for geometric NP-hard problems

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Reading Group, May 17, 2011

max planck institut
informatik

or

On Euclidean vehicle routing with allocation

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The Traveling Salesman Problem

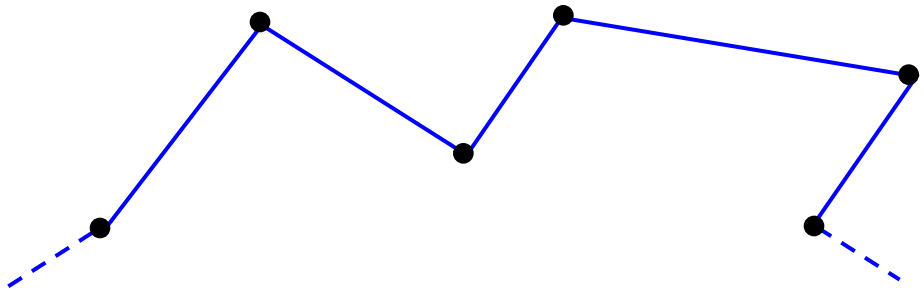
- **The Traveling Salesman Problem (TSP)**
  - Input: edge-weighted graph G
  - Output: Hamilton cycle in G with minimum edge-weight
- **Motivation:**
  - Traveling salesman ;-)
- **Complexity:**
  - NP-hard
  - Admits no constant factor approximation (unless P=NP) [Sahni and Gonzalez 76]

Metric TSP

- **Metric TSP**
  - Input: edge-weighted graph G satisfying triangle inequality
  - Output: Hamilton cycle in G with minimum edge-weight
- **Motivation:**
  - real-world problems usually satisfy triangle inequality
- **Complexity:**
  - still NP-hard
  - admits 3/2-approximation [Christofides 76]
  - admits no PTAS (unless P=NP) [Arora *et al.* 98]

Euclidean TSP

- **Euclidean TSP**
  - Input: points $P \subset \mathbb{R}^2$
  - Output: tour π through P with minimal length
- **Complexity:**
  - still NP-hard [Papadimitriou 77]
  - admits PTAS [Arora 96; Mitchell 96]



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  - admits PTAS [Arora 96; Mitchell 96]

Arora (FOCS '97)

There is a randomized PTAS for Euclidean TSP with complexity $n \log^{O(1/\epsilon)} n$.

- …even one with complexity O(n log n).

Rao, Smith (STOC '98)

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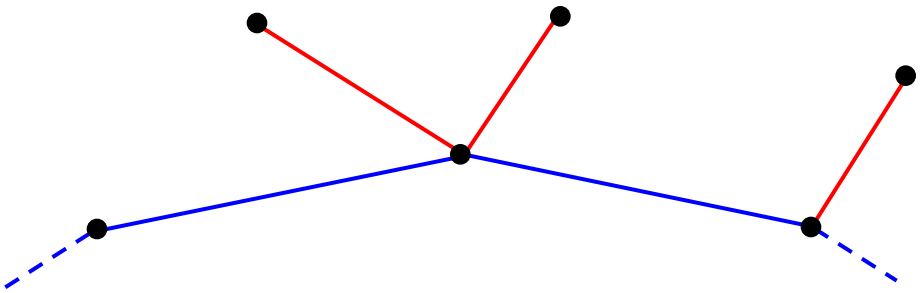
VRAP

- **(Euclidean) Vehicle Routing with Allocation (VRAP)**
  - Input: points $P \subset \mathbb{R}^2$, constant $\beta \geq 1$
  - Output: tour π through subset $T \subseteq P$ minimizing

$$\underbrace{\sum_{\{p,q\} \in \pi} d(p,q)}_{\text{tour length}} + \underbrace{\beta \cdot \sum_{p \in P \setminus T} \min_{q \in T} d(p,q)}_{\text{allocation cost}}$$

- **Motivation:**
  - salesman does not visit all customers
  - customers not visited go to next tourpoint, which is more expensive by a factor of β.



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- **Complexity:**
  - NP-hard, since setting $\beta \geq 2$ yields Euclidean TSP
  - as for Euclidean TSP, there exists a quasilinear PTAS

Remy, S., Weißl (WADS '07)

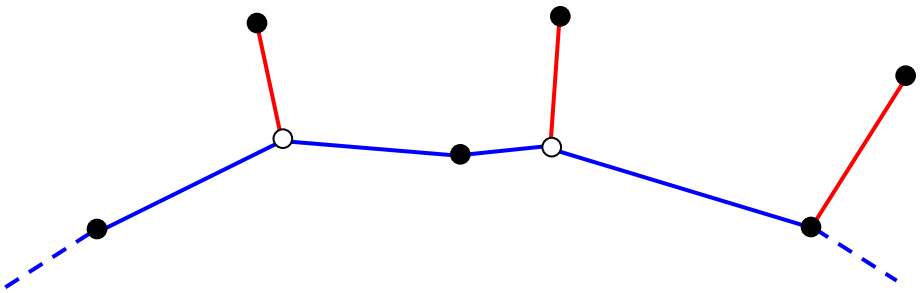
There is a randomized PTAS for VRAP with complexity $O(n \log^4 n)$.

Steiner VRAP

- **Steiner VRAP**
  - Input: points P $\subset \mathbb{R}^2$, constant $\beta \geq 1$
  - Output: subset T $\subseteq$ P, set of points S $\subset \mathbb{R}^2$ (Steiner Points), tour π through T $\cup$ S minimizing

$$\underbrace{\sum_{\{p,q\}\in\pi} d(p,q)}_{\text{tour length}} + \underbrace{\beta \cdot \sum_{p\in P\setminus T} \min_{q\in T\cup S} d(p,q)}_{\text{allocation cost}}$$

- **Motivation:**
  - salesman may also stop en route to serve customers



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- **Complexity:**
  - NP-hard
  - admits PTAS

Armon, Avidor, Schwartz (ESA '06)

There is a randomized PTAS for Steiner VRAP with complexity $n^{O(1/\epsilon)}$.

  - …even a quasilinear one

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There is a randomized PTAS for Steiner VRAP with complexity $n \log^{O(1/\epsilon)} n$.

Techniques

- Finding a good solution for VRAP means
  - a) finding a good set of tour points $T \subseteq P$
  - b) finding a good tour on this set T

  simultaneously.
- **a)** is essentially a facility location problem.
  - We use the adaptive dissection technique, due to [Kolliopoulos and Rao, ESA '99]
- **b)** is Euclidean TSP.
  - We use dynamic programming on 'patched short spanners', due to [Rao and Smith, STOC '98]
- To put both ideas into perspective, we start by explaining the basics of dynamic programming in quadtrees, as introduced in [Arora, FOCS '96] for Euclidean TSP

3

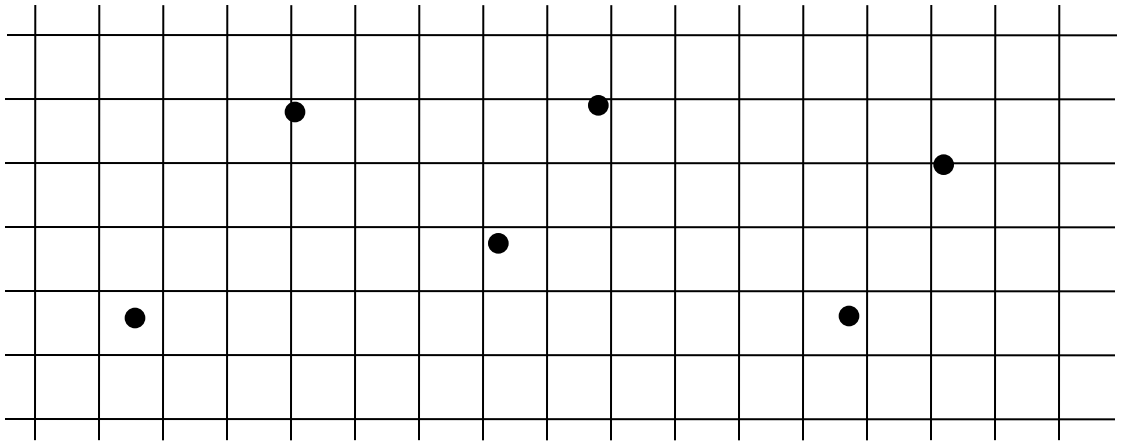
2

1

Preliminaries

- We assume that the input points P
  - have odd integer coordinates
  - lie inside a square whose sidelength is
    - a power of 2
    - of order O(n/ϵ)
- This is ok, since every (1+ϵ/2)-approximation for the rescaled and shifted input P' corresponds to a (1+ϵ)-approximation for the original input P.

P



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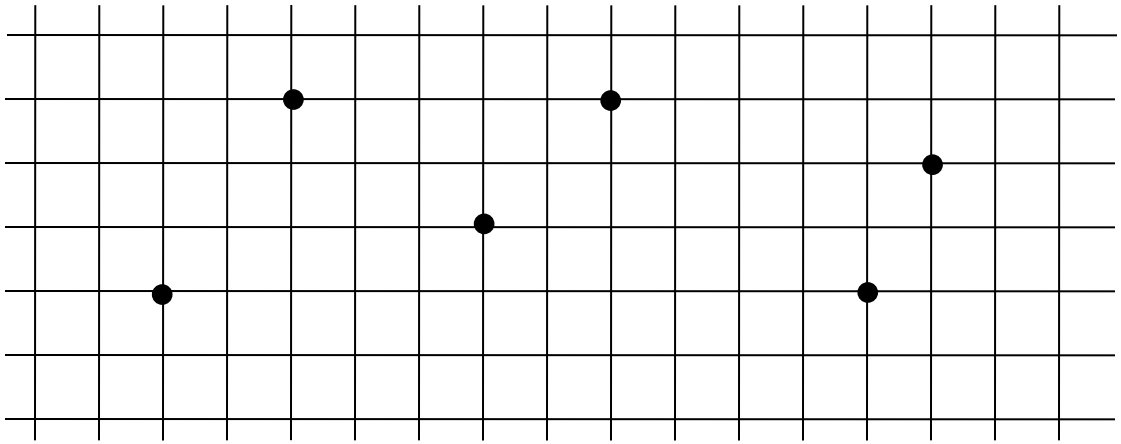

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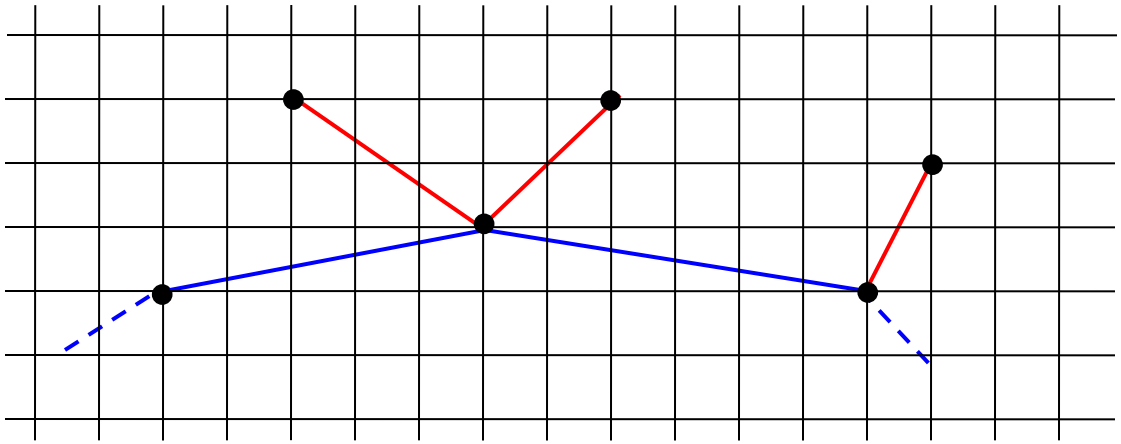

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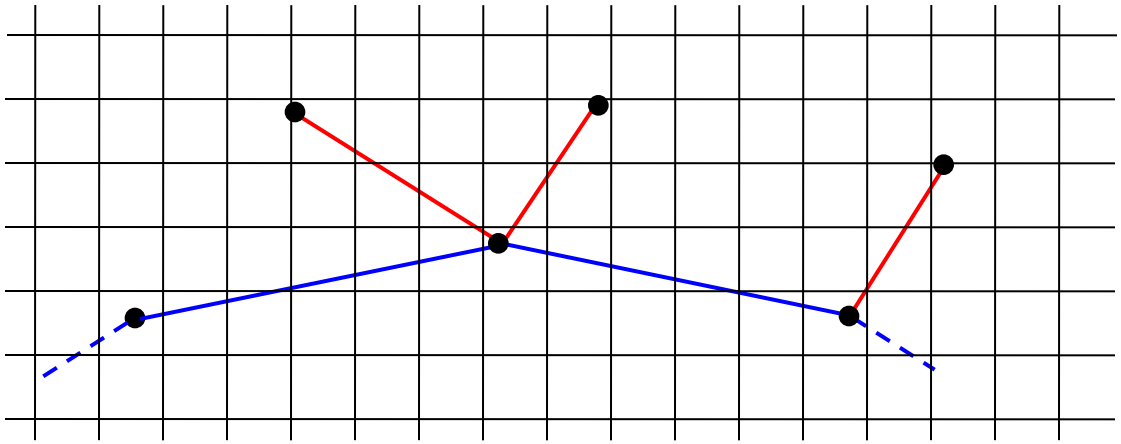
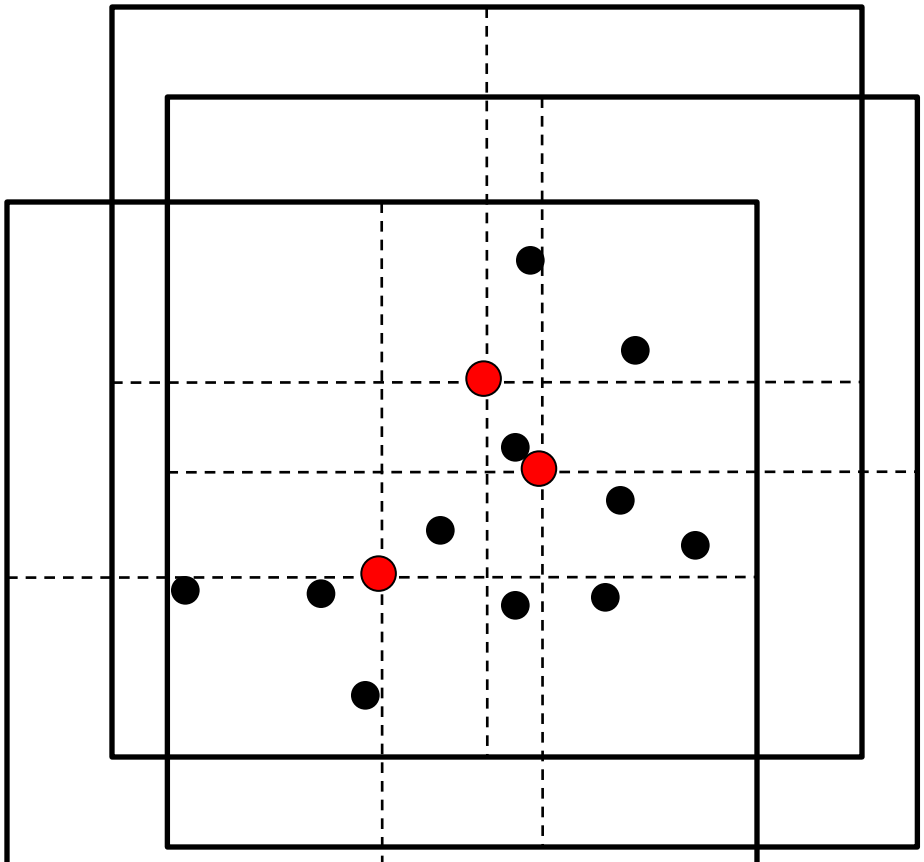


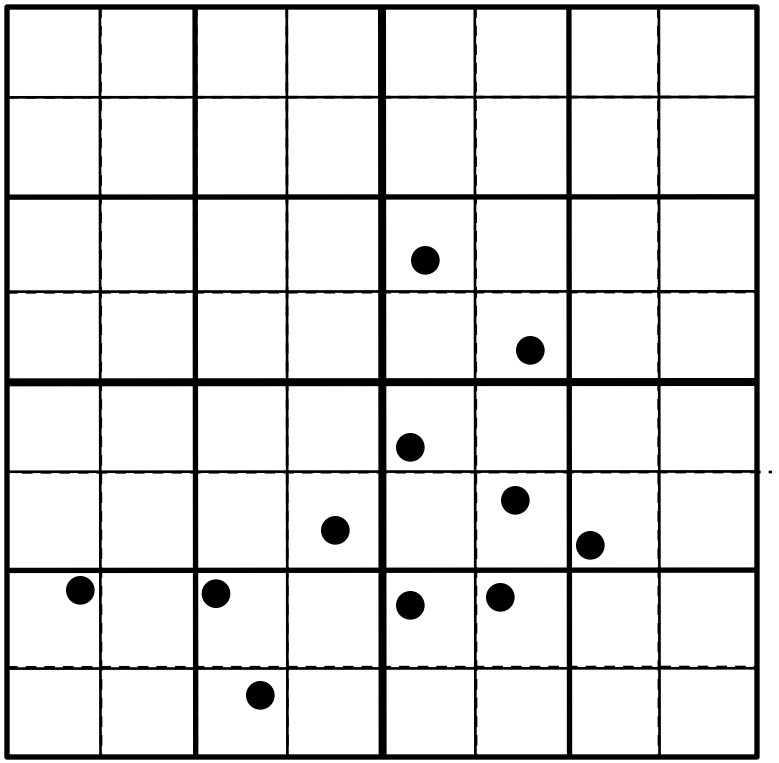
Quadtrees

- Choose origin of coordinate system (= center of large square) randomly.
  - this is the only source of randomness in all algorithms



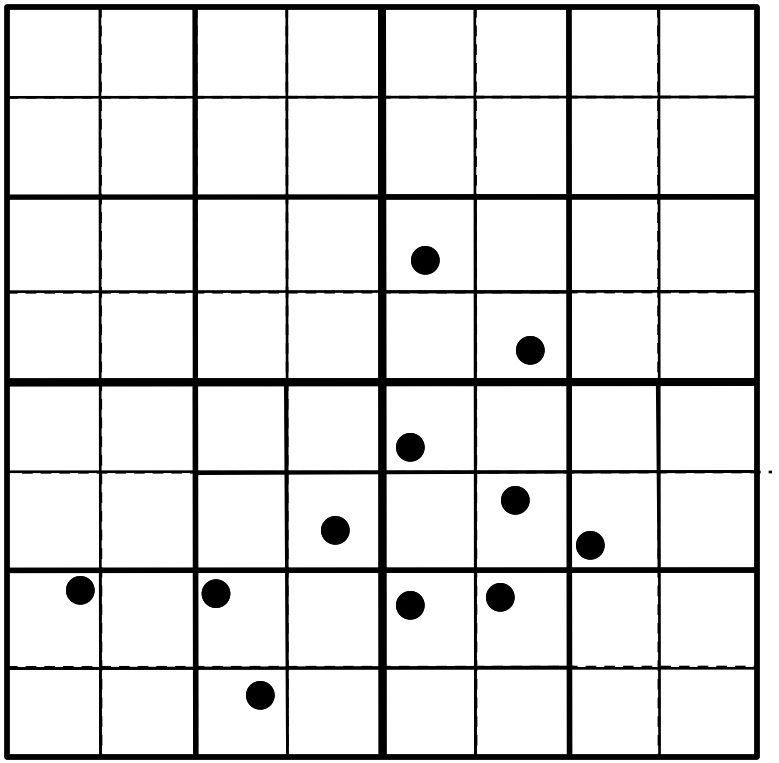
Quadtrees

- Split large square recursively into 4 smaller squares until squares have sidelength 2
  - Since bounding square has sidelength O(n), resulting tree has $O(n^2)$ nodes (squares) and depth O(log n)



Quadtrees

- Truncated quadtree: stop subdivision at empty squares
  - remaining tree has O(n log n) nodes

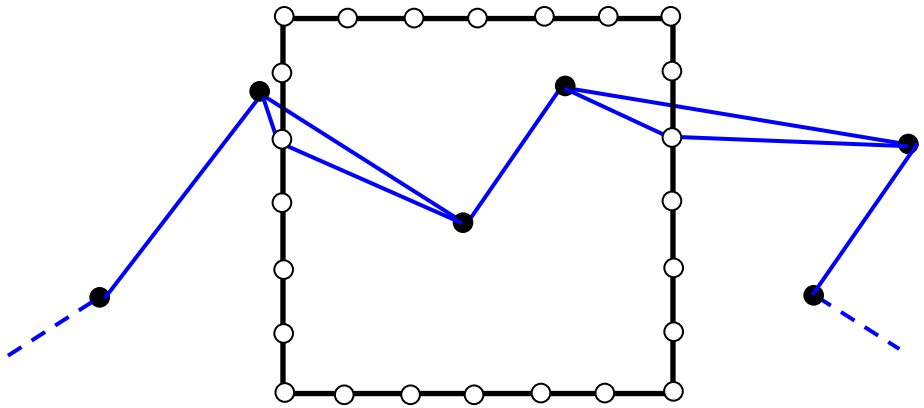


Portal-respecting solutions

- Place O(log n/ϵ) many equidistant points (‘portals’) on the boundary of each square.
  - **Impose restriction**: Salesman may enter/leave a square only via its portals.

Lemma (Arora)

In expectation, detouring all edges of the optimal salesman tour via the nearest portal increases its length only by a factor of 1+ϵ.



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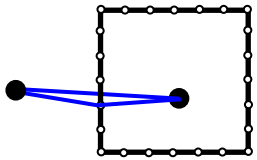
- **Intuition:** for two fixed points:
  - good



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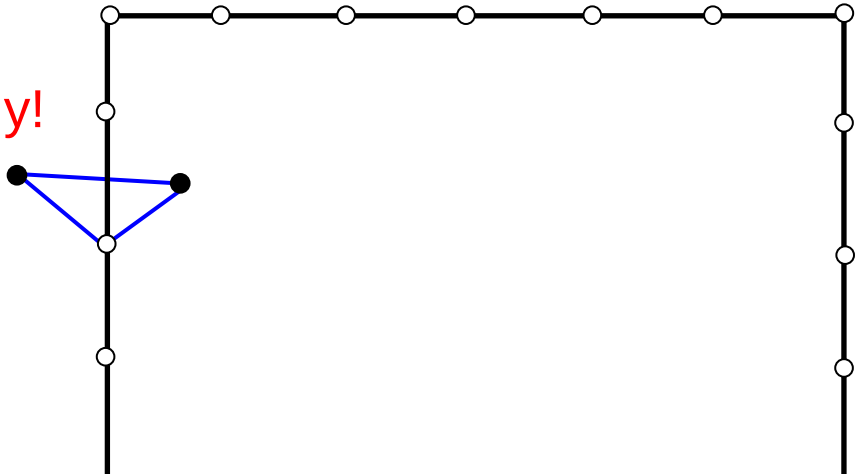
- **Intuition:** for two fixed points:
  - bad
  - but unlikely!



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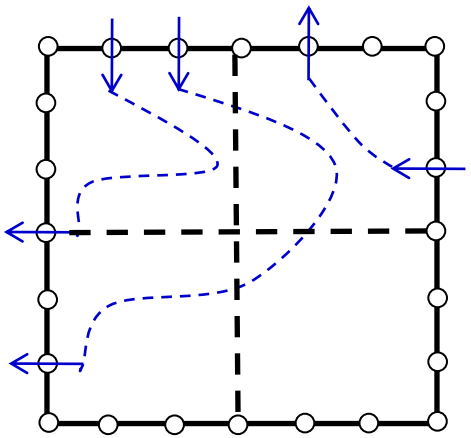
In expectation, detouring all edges of the optimal salesman tour via the nearest portal increases its length only by a factor of 1+ϵ.

  - i.e., there is an expected nearly-optimal portal-respecting salesman tour.
- We try to find the best portal-respecting salesman tour by dynamic programming in the quadtree.

Dynamic programming in quadtrees

- For a given square Q, guess which portals are used by salesman tour, and enumerate all possible configurations $\mathcal{C}$.
- For each configuration $\mathcal{C}$, calculate estimate for the length of a good tour inside Q, subject to the restrictions given by $\mathcal{C}$:
  - If Q is a leaf of the quadtree, by brute force.
  - If Q is an inner node of the quadtree, by recursing to its four children.

$\mathcal{C}$



Running time

- Working in a non-truncated quadtree, we have to consider $O(n^2)$ squares. For each of these we have to consider $2^{O(\log n/\epsilon)} = n^{O(1/\epsilon)}$ configurations, and the estimate for each configuration can be calculated in time $n^{O(1/\epsilon)}$.
  - We obtain a PTAS with running time

$$O(n^2) \cdot n^{O(1/\epsilon)} \cdot n^{O(1/\epsilon)} = \underline{\underline{n^{O(1/\epsilon)}}}$$

Arora (FOCS '96)

There is a randomized PTAS for Euclidean TSP with complexity $n^{O(1/\epsilon)}$.

- This is essentially the technique used in the PTAS for Steiner VRAP by Armon *et al*.

Armon, Avidor, Schwartz (ESA '06)

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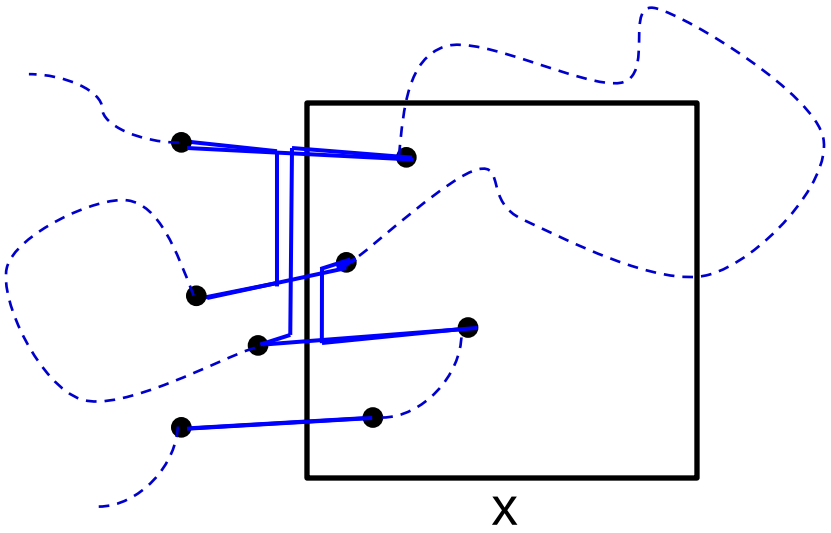
- to achieve quasilinear time, we can only use polylogarithmic time per square. In particular, we can only consider polylogarithmically many configurations per square.

Improving the running time

Patching Lemma (Arora)

The optimal solution can be modified such that it crosses the boundary of every square at most $O(1/\epsilon)$ many times.
In expectation, this increases the length of the tour only by a factor of $1+\epsilon$.

- **Idea:** proceed bottom-up through quadtree and modify each square with too many crossings by introducing line segments parallel to sides.



- The total length of the new line segments is at most 3x
- ➔ modification on low levels of the quadtree are cheap.

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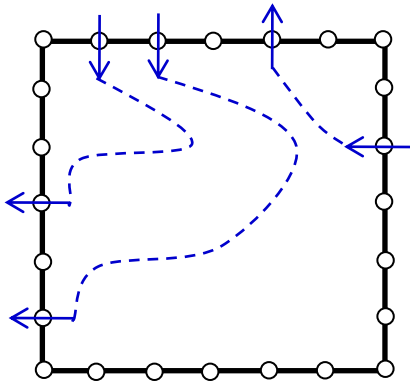
In expectation, this increases the length of the tour only by a factor of $1+\epsilon$.

- i.e., there is an expected nearly-optimal portal-respecting salesman tour which for every square uses only $O(1/\epsilon)$ many of the $O(\log n)$ portals.

- Looking for such a 'patched' solution, we only have to consider $O(\log n)^{O(1/\epsilon)} = \log^{O(1/\epsilon)} n$ configurations per square!



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In expectation, this increases the length of the tour only by a factor of $1+\epsilon$.

- We only have to consider $\log^{O(1/\epsilon)} n$ configurations per square.
  - Working in a truncated quadtree, we obtain a PTAS with running time

$$O(n \log n) \cdot \log^{O(1/\epsilon)} n \cdot \log^{O(1/\epsilon)} n = \underline{\underline{n \log^{O(1/\epsilon)} n}}$$

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In expectation, this increases the length of the tour only by a factor of $1+\epsilon$.

Lemma

The Patching Lemma extends to Steiner VRAP.

- Combining the extended patching lemma with standard quadtree techniques for facility location problems [Arora, Raghavan, Rao, STOC '98], we obtain

Remy, S., Weißl (WADS '07)

There is a randomized PTAS for Steiner VRAP with complexity $n \log^{O(1/\epsilon)} n$.

Improving the running time even further

- **Patching revisited:**
  - In Arora's technique, the 'patching' is not part of the algorithm – we simply know a nearly-optimal patched solution exists, and try to find it by dynamic programming.
  - Rao and Smith (STOC '98) improved Arora's running time by making the 'patching' part of the algorithm.
- A **(1+ϵ)-spanner S** on P is a straight-line graph on P such that for every two points the shortest path in S is at most (1+ϵ) time their Euclidean distance.
  - A 'short' (1+ϵ)-spanner can be computed in time O(n log n) [Gudmundsson, Levcopoulos, Narasimhan, SWAT '00]
- Clearly, given such a spanner S there is a nearly-optimal salesman tour that only uses edges of S.

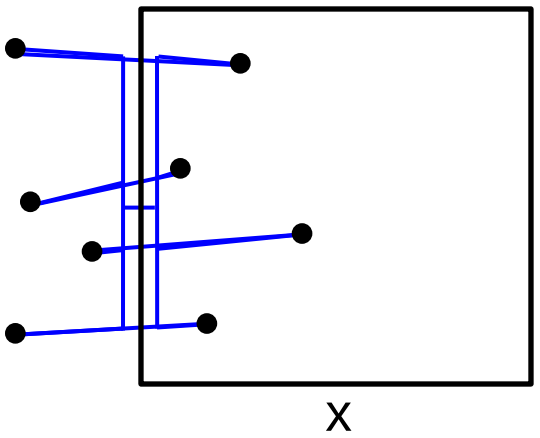
Improving the running time even further

Patching Lemma (Rao and Smith)

A short spanner S can be modified such that every square of the quadtree is crossed by at most $O(1/\epsilon)$ relevant edges.

In expectation, this increases the length of an optimal tour on the graph only by a factor of $1+\epsilon$.

- **Idea:** proceed bottom-up through quadtree and modify each square with too many crossings by introducing line segments parallel to sides.



- The total length of the new line segments is at most 2x
- ➔ modification on low levels of the quadtree are cheap.

Improving the running time even further

- **A better algorithm for Euclidean TSP**:
  - Compute short spanner S on P
  - Patch S, call the new graph S'
  - Dynamic programming in quadtree, but instead of portals use edges of S'.
- We now only have to consider constantly many configurations per square!
  - We obtain a PTAS with running time

$$O(n \log n) \cdot O(1) \cdot O(1) = \underline{\underline{O(n \log n)}}$$

Rao, Smith (STOC '98)

There is a randomized PTAS for Euclidean TSP with complexity O(n log n).

Dealing with the facility location part

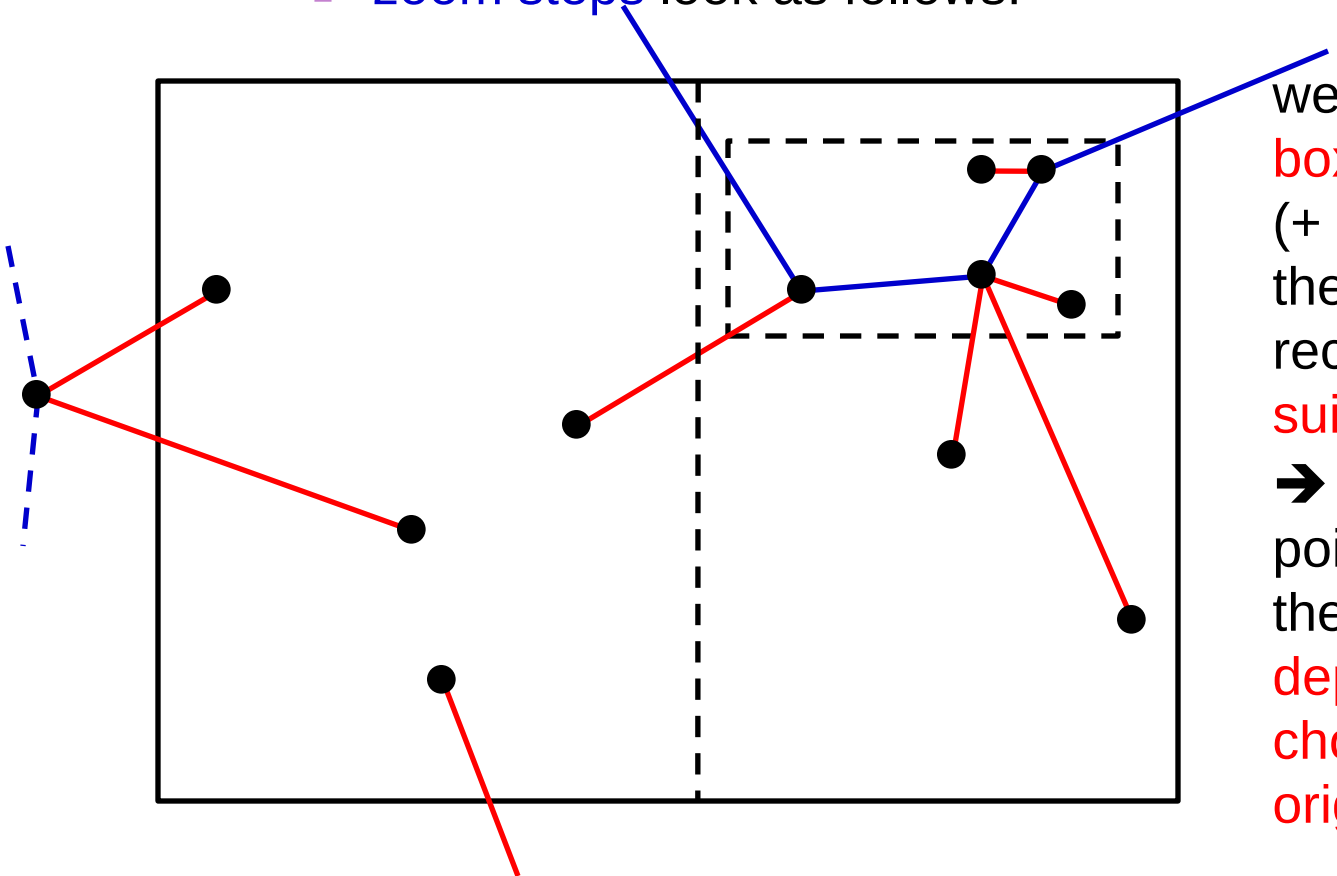
- I promised a running time of $O(n \log^4 n)$ for (non-Steiner) VRAP.
- The techniques discussed so far take care of the Euclidean TSP part of the problem, and it remains to discuss the facility location part.
- I will present the key ideas directly for the VRAP setting.

Adaptive dissection

- To improve the running time for facility location problems, Kolliopoulos and Rao (ESA '99) introduced the adaptive dissection technique:
  - the quadtree is replaced by a more complicated structure, a zoom tree
  - the structure of the zoom tree changes with the location of the facilities (in our case, the tour points T).
    - Guessing the location of the tour points is done by guessing how to best recurse.
    - we have to do dynamic programming in larger structure, which is essentially the union of the zoom trees for all possible choices of $T \subseteq P$
- **Key Advantage:** constantly many portals per rectangle suffice!
- Everything discussed so far (for the 'TSP part') needs to be adapted from the quadtree setting to the zoom tree setting!

The zoom tree

- The zoom tree alternates between split steps and zoom steps.
  - split steps work very similar to recursion in quadtree.
  - zoom steps look as follows:



we zoom on bounding box of tour points (+ some safety margin), the sides of this rectangle lie on a suitable grid.

➔ for a fixed set of tour points, the structure of the resulting zoom tree depends on the random choice of the coordinate origin.

How does this help?

- Two conceptual advantages:
  - On one hand, directly zooming on the tour points skips levels in between, which might introduce large errors in the quadtree technique.
  - On the other hand, in the resulting nearly-optimal solution, a point is not necessarily allocated to its nearest tourpoint, but possibly to a different nearby point. ➔ added flexibility in analysis.

- The net effect is that we only have to consider constantly many configurations per rectangle.

Final running time for VRAP

- Running time is dominated by zoom steps
  - We consider rectangles of bounded aspect ratio with sides on suitable grids containing at least one point.
    - ➔ There are only $O(n \log^2 n)$ pairs of rectangles which correspond to zoom steps
  - For each such pair, the zoom step can be performed in time $O(\log^2 n)$.
    - This requires allocating non-tour points in batches using range searching techniques.
  - We obtain a running time of

$$O(n \log^2 n) \cdot O(1) \cdot O(\log^2 n) = \underline{\underline{O(n \log^4 n)}}$$

Remy, S., Weißl (WADS '07)

There is a randomized PTAS for VRAP with complexity $O(n \log^4 n)$.

Higher dimensions

- Arora's PTAS for Euclidean TSP and our PTAS for Steiner VRAP extend to any fixed dimension d, yielding a running time of $O(n \log^{C(d,\epsilon)} n)$.
  - Here $C(d, \epsilon) = O(\sqrt{d}/\epsilon)^{d-1}$, i.e., the running time is doubly exponential in d.
  - **Main difficulty**: the patching becomes more complicated, since the 'sides' of the hyper-'squares' are now (d–1)-dimensional hypercubes.

Higher dimensions

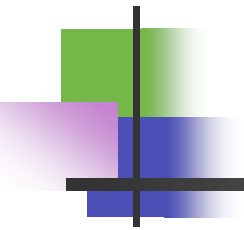
- Arora's PTAS for Euclidean TSP and our PTAS for Steiner VRAP extend to any fixed dimension d, yielding a running time of $O(n \log^{C(d,\epsilon)} n)$.

- The Rao-Smith PTAS for Euclidean TSP extends to any fixed dimension d, still having a running time of $O(n \log n)$.
  - (but with the implicit constant depending badly on d)
- Our PTAS for VRAP extends to any fixed dimension d, yielding a running time of $O(n \log^{d+2} n)$.
  - The range searching adds an extra log-factor per dimension.
- All algorithms can be derandomized by enumerating all possible random shifts of the quadtree (zoom tree), at the cost of an extra factor $O(n^d)$.

Summary

- VRAP is a combination of Euclidean TSP and a facility location problem.
- The two state-of-the-art techniques
  - Dynamic programming on ‘patched short spanners’ (Rao and Smith, STOC ’98) for Euclidean TSP
  - Adaptive dissection (Kolliopoulos and Rao, ESA ’99) for facility location

can be combined into a $O(n \log^4 n)$-PTAS for VRAP.



Thank you!
Questions?

Proof of Lemma

- Consider two fixed points **u**,**v** $\in$ P, and a fixed line **g** which is on the even-integer grid and intersects **uv**.
- If **g** is part of the level in which the squares have sidelength s, it introduces an error of at most s/m at this level.
- The probability that **g** is part of the level in which the squares have sidelength s is 2/s.
- That is, the grid line **g** introduces an expected error of at most

$$2/s \cdot s/m = 2/m$$

  at each level**,** which is O(log n/m) in total.
- Since there are at most d(u,v) even-integer grid lines intersecting **uv**, the total expected error in the distance from **u** to **v** is

$$O(\log n/m) \cdot d(u,v) = O(\epsilon) \cdot d(u,v) .$$

- increase number of portals by constant factor to beat constant hiding in O ➔ in expectation, tour length increases by a factor of $1+\epsilon$.