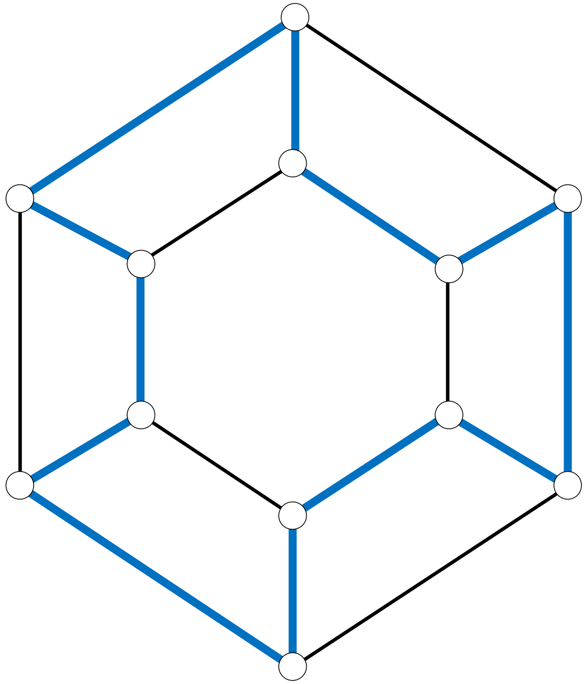
On Barnette's Conjecture

Jens M. Schmidt

Hamiltonian Cycles

Def. A graph is Hamiltonian if it contains a Hamiltonian cycle, i.e., a cycle that contains every vertex exactly once.

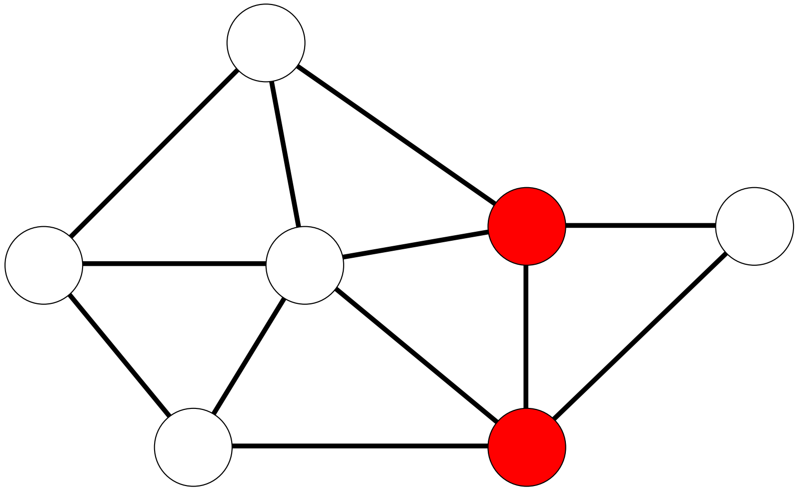


William R. Hamilton

3-Connectivity

Let $G=(V,E)$ be a simple finite graph, $n=|V|$, $m=|E|$.

Def. G is 3-connected $\Leftrightarrow$ $n > 3$ and there is no separation pair in G



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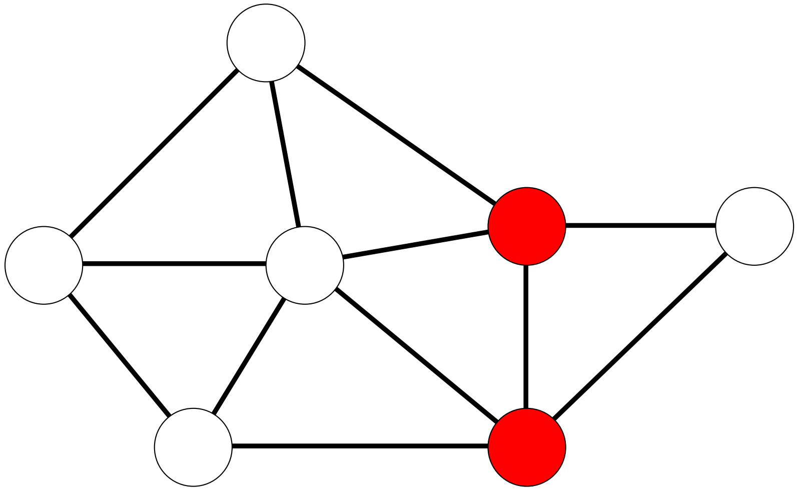
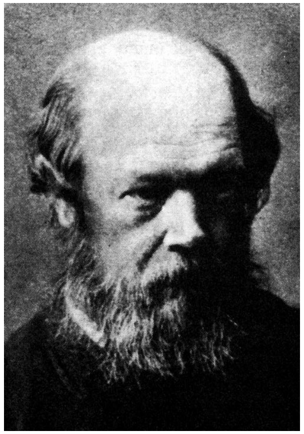


<u>Thm (Steinitz 1922).</u> The graphs of convex 3-dimensional polyhedra are exactly the (simple) planar 3-connected graphs.

Tait's Conjecture

Tait's Conjecture (1884). Every cubic planar 3-connected graph is Hamiltonian.



Peter G. Tait

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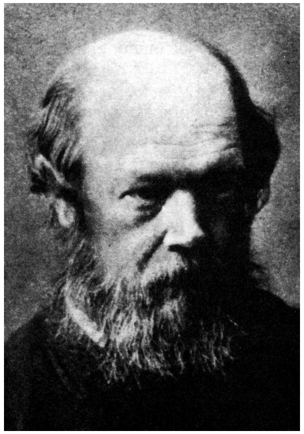

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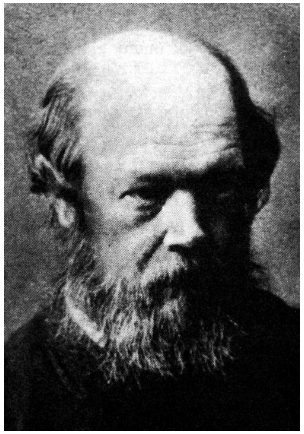
would imply the 4-color theorem



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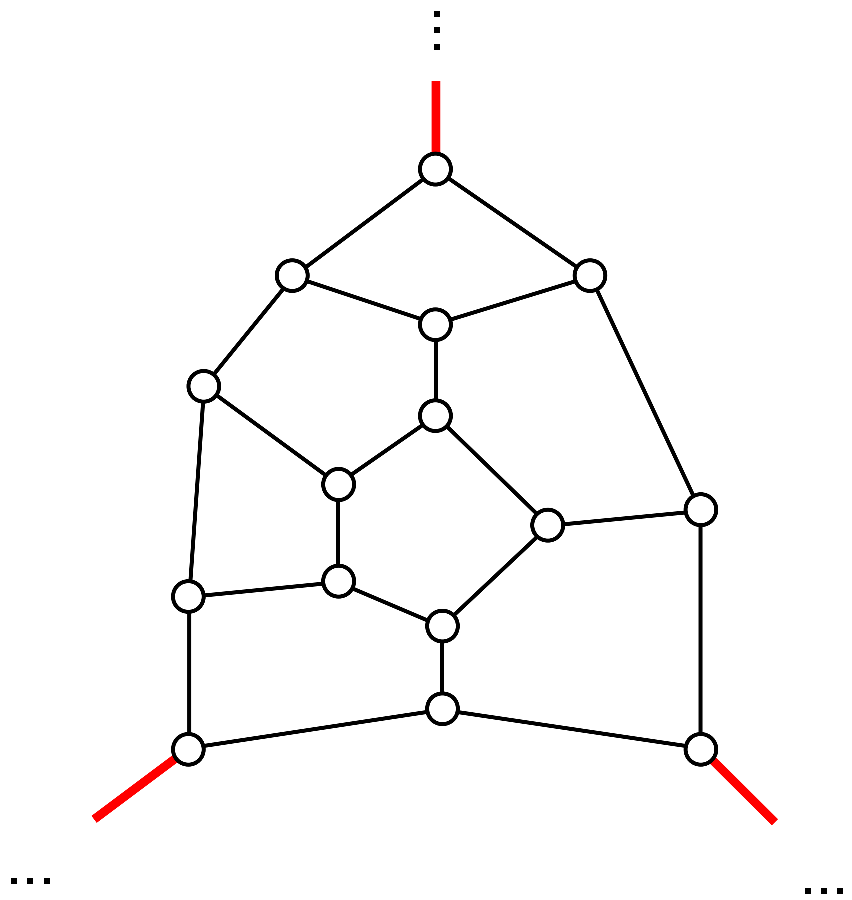

Tutte's "fragment"

William T. Tutte

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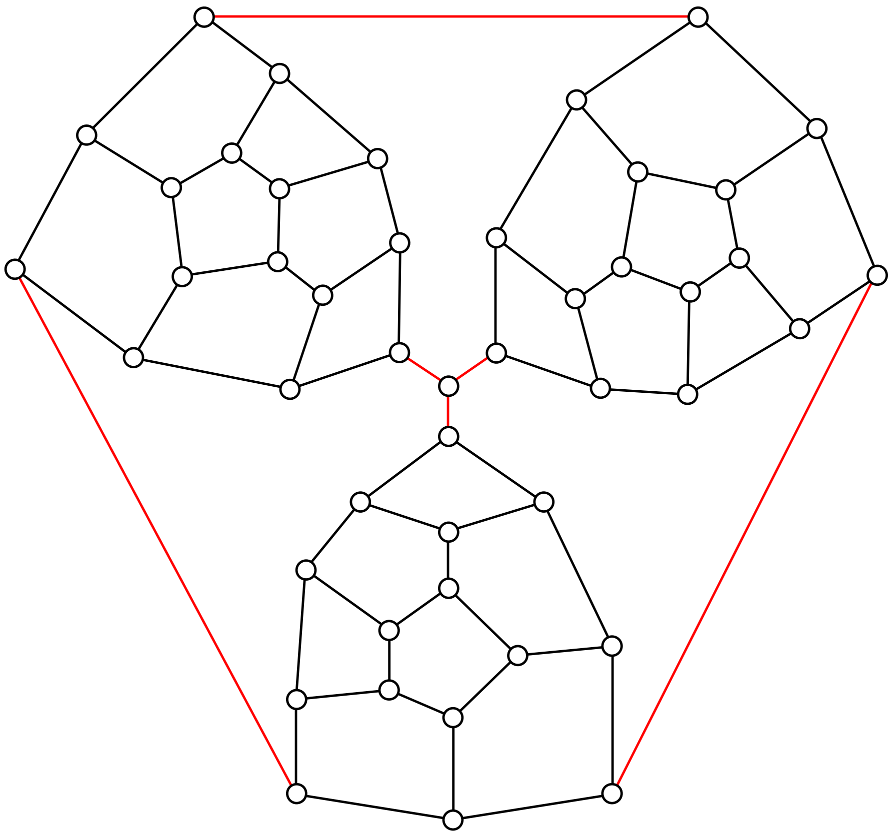

Tutte's counterexample (1946, 46 vertices)

William T. Tutte

Tutte's Conjecture

~~planar~~

Tutte's Conjecture (1971). Every cubic bipartite 3-connected graph is Hamiltonian.

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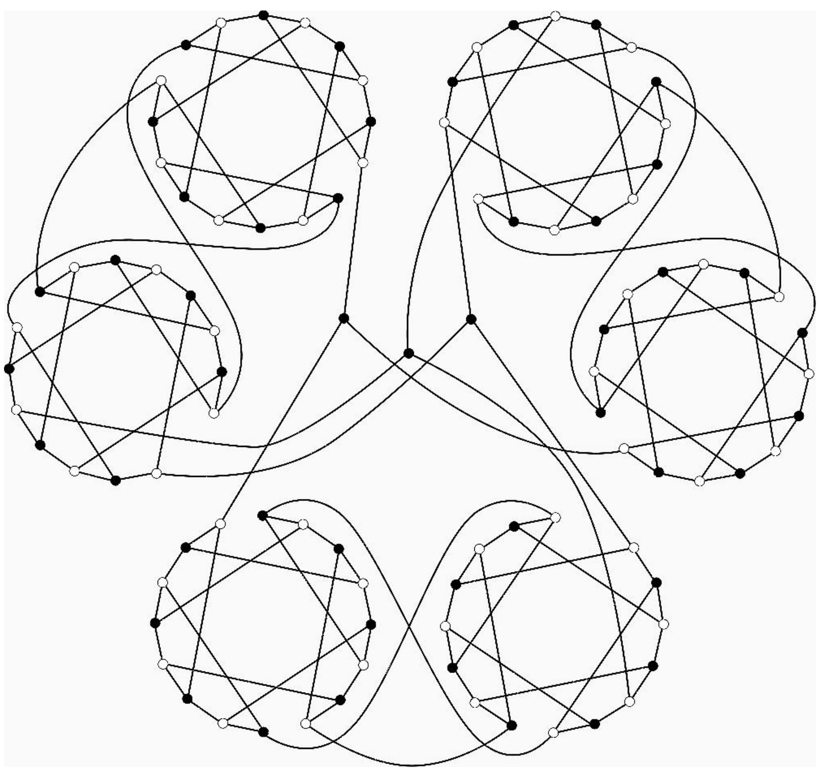

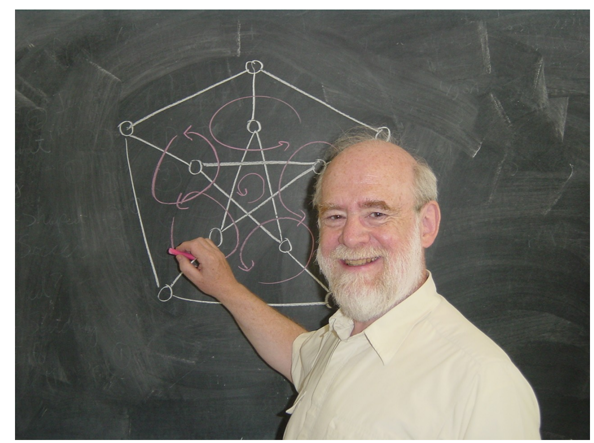

Joseph D. Horton

Counterexamples of Baraev & Faradzhev (1978) and Horton (1982)

Barnette's Conjecture

Barnette's Conjecture (1969). Every cubic bipartite planar 3-connected graph is Hamiltonian.

David W. Barnette

Barnette's Conjecture

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still open...

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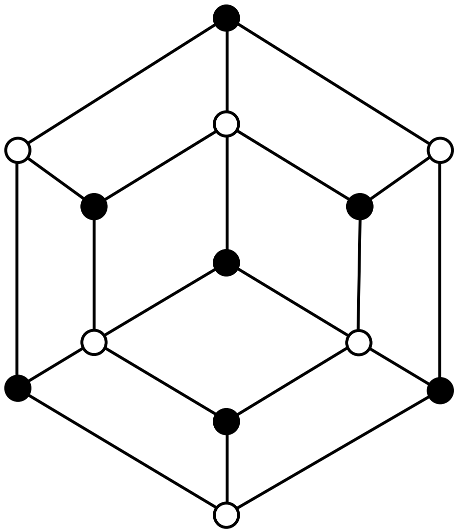

Kirkman graph

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“Barnette graph”

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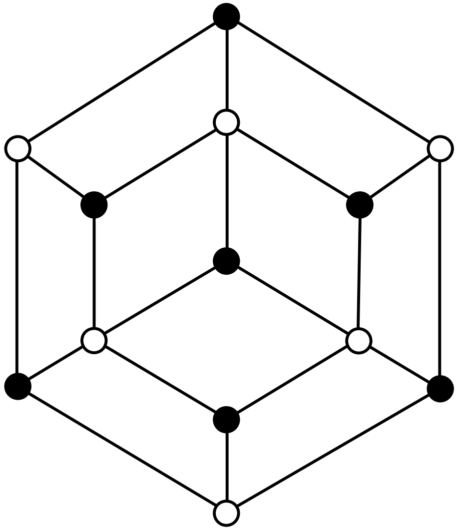

Kirkman graph

Strengthening the Conjecture

A Barnette graph is x-$\bar{y}$-Hamiltonian if for every two edges *x* and *y* in a common face *f* there is a Hamiltonian cycle that contains *x* but not *y*.

Thm (Kelmans 1986). Barnette's conjecture is true if and only if every Barnette graph is x-$\bar{y}$-Hamiltonian.

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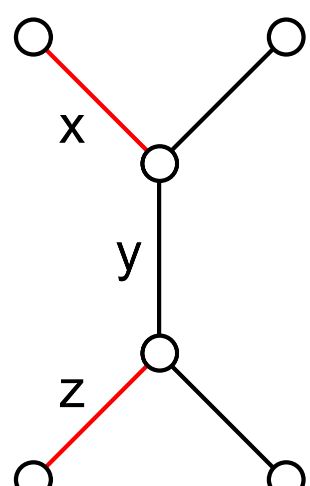

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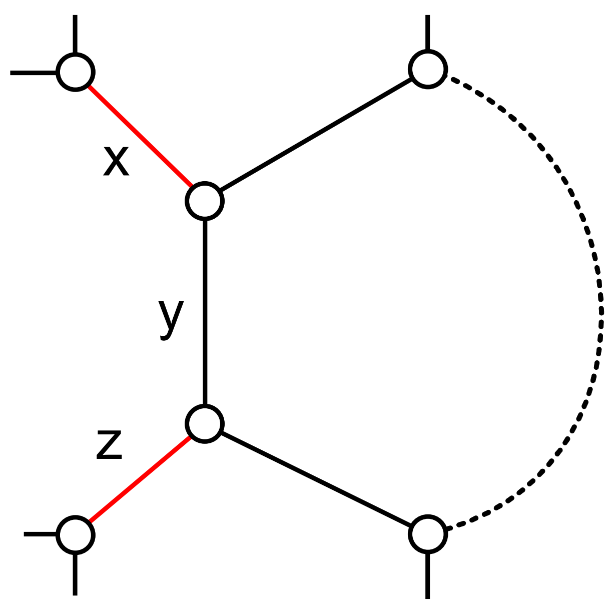

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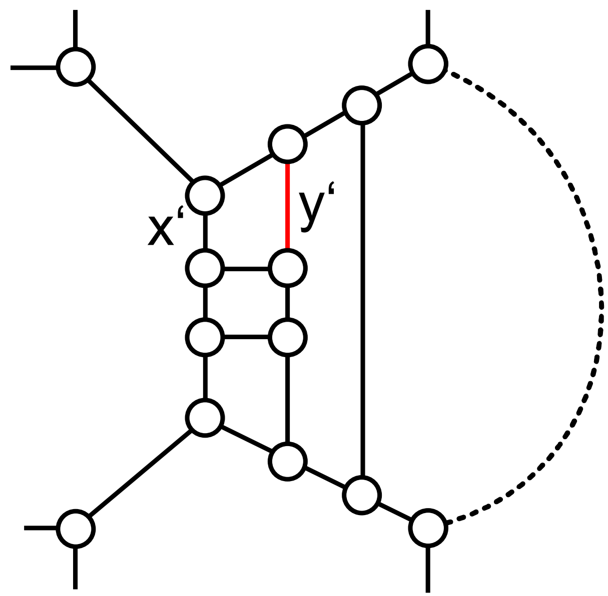

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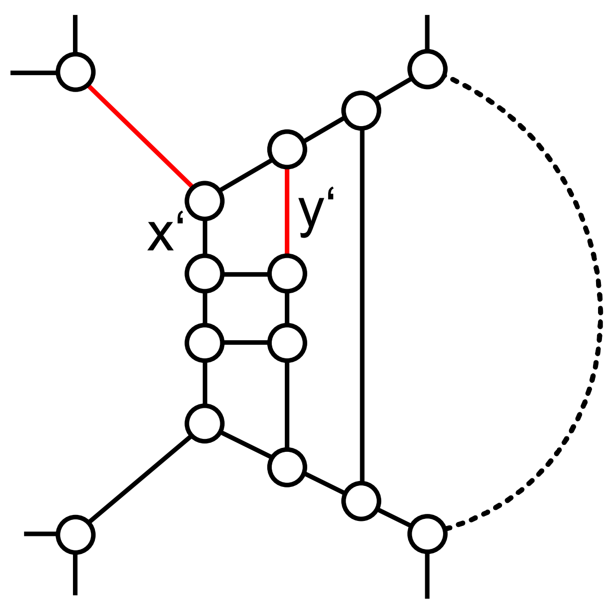

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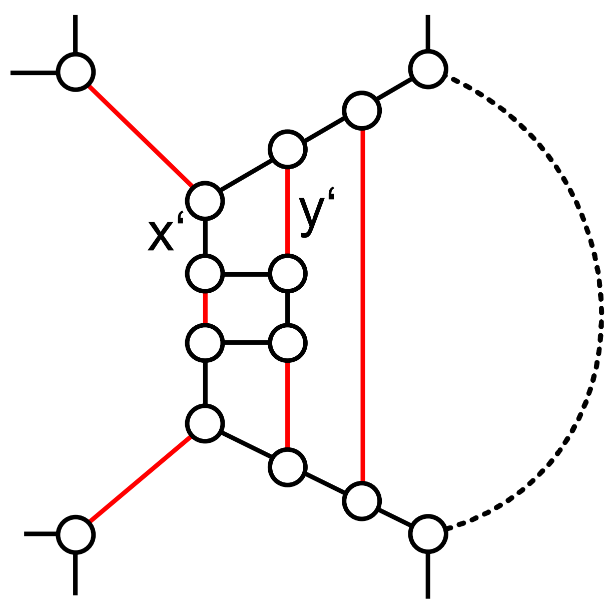

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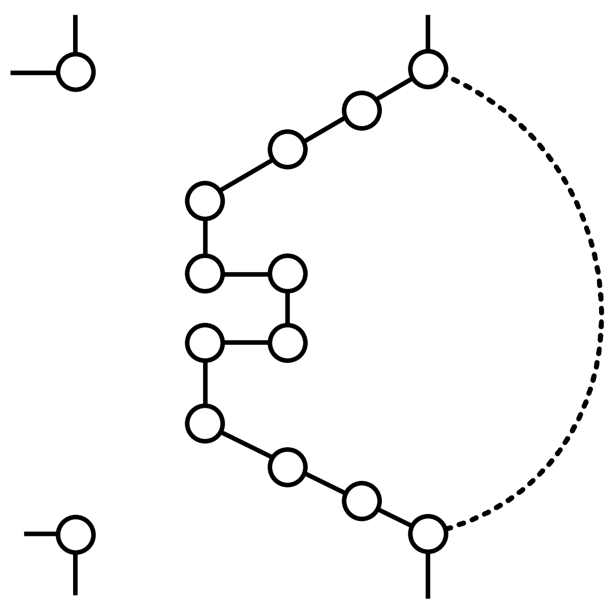

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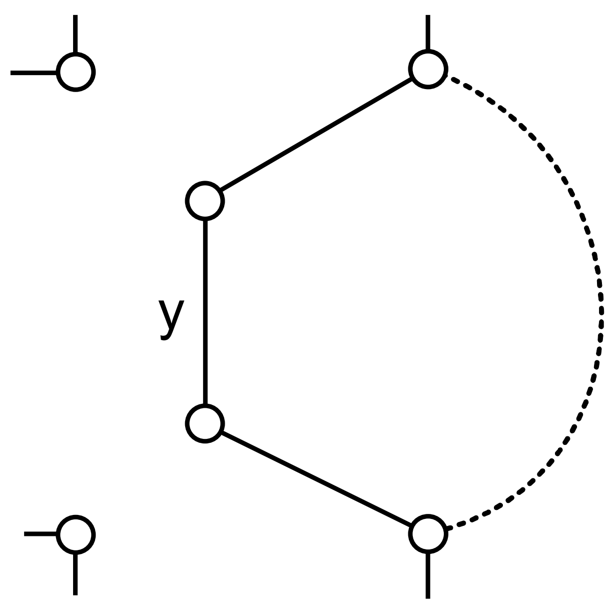


Dualization & Computer-aided Search

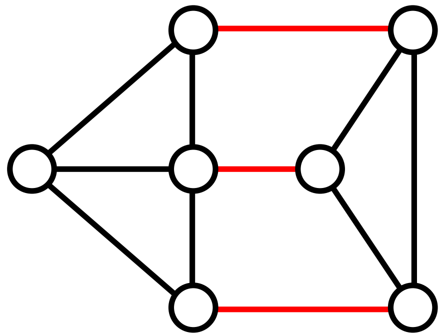
Searching for counterexamples:

Thm (Brinkmann, McKay 2002). Every Barnette graph with at most 84 vertices is Hamiltonian.

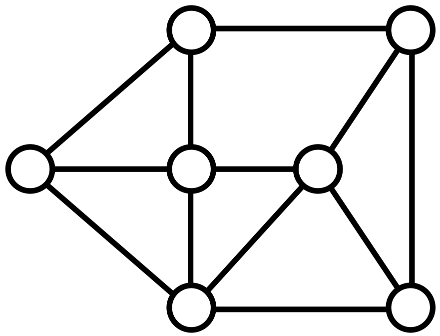
Thm (Brinkmann, McKay 2002). Every Barnette graph with at most 60 vertices is x-$\bar{y}$-Hamiltonian. Even true if *x* and *y* are not necessarily in the same face.

Weakening the Conjecture

Def. A graph is cyclically 4-edge-connected if every cycle-separating edge cut contains at least 4 edges.



not cyclically 4-edge-connected



cyclically 4-edge-connected

Weakening the Conjecture

<u>Thm (S.).</u> Barnette's conjecture is true if and only if every cyclically 4-edge-connected Barnette graph is x-$\bar{y}$-Hamiltonian.

<u>Proof.</u>

- =>: easy.

Weakening the Conjecture

Thm (S.). Barnette's conjecture is true if and only if every cyclically 4-edge-connected Barnette graph is x-$\bar{y}$-Hamiltonian.

Proof.

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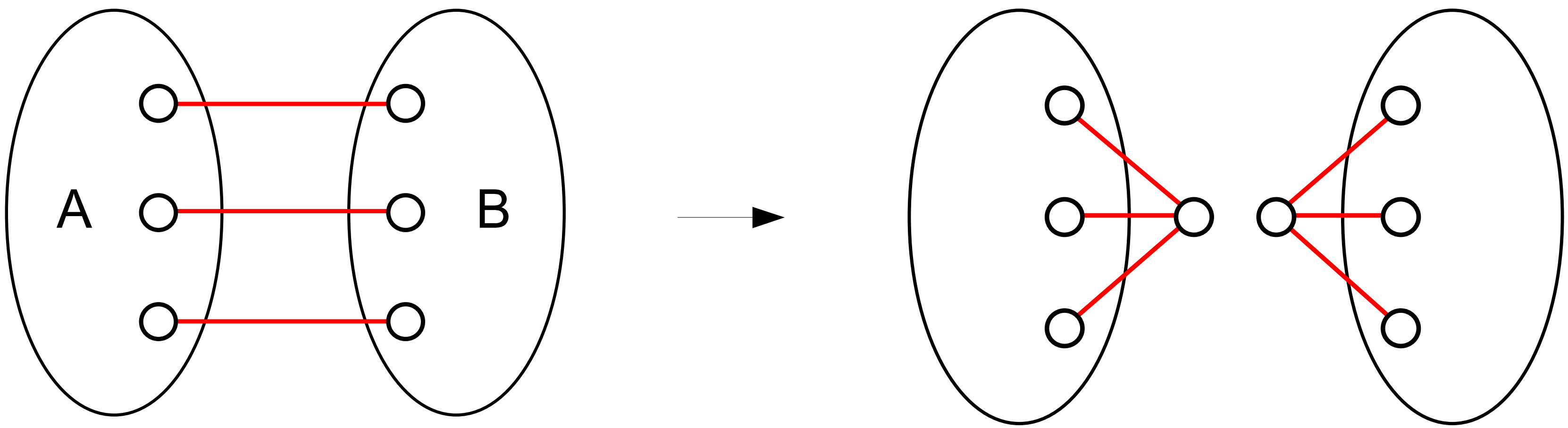
- <=: Take any non-cyclically 4-edge-connected Barnette graph.



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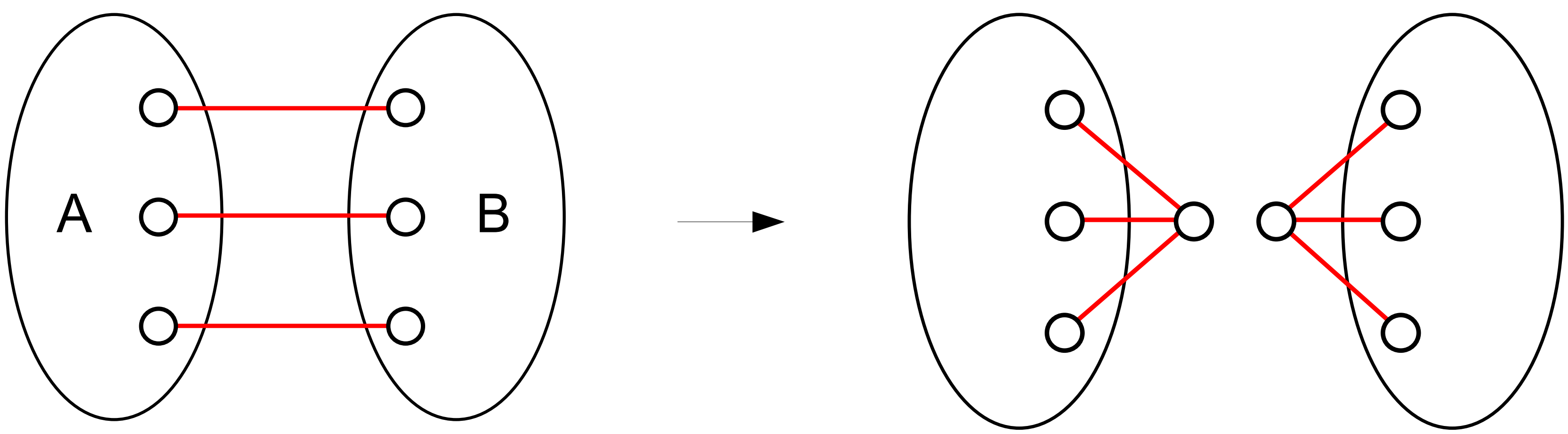

- cubic

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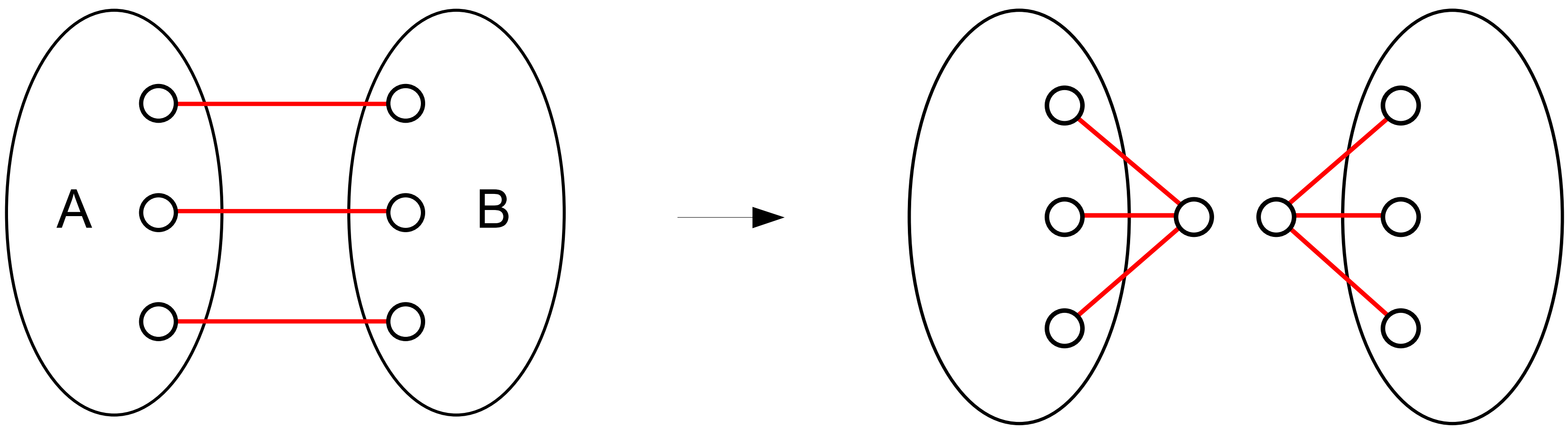

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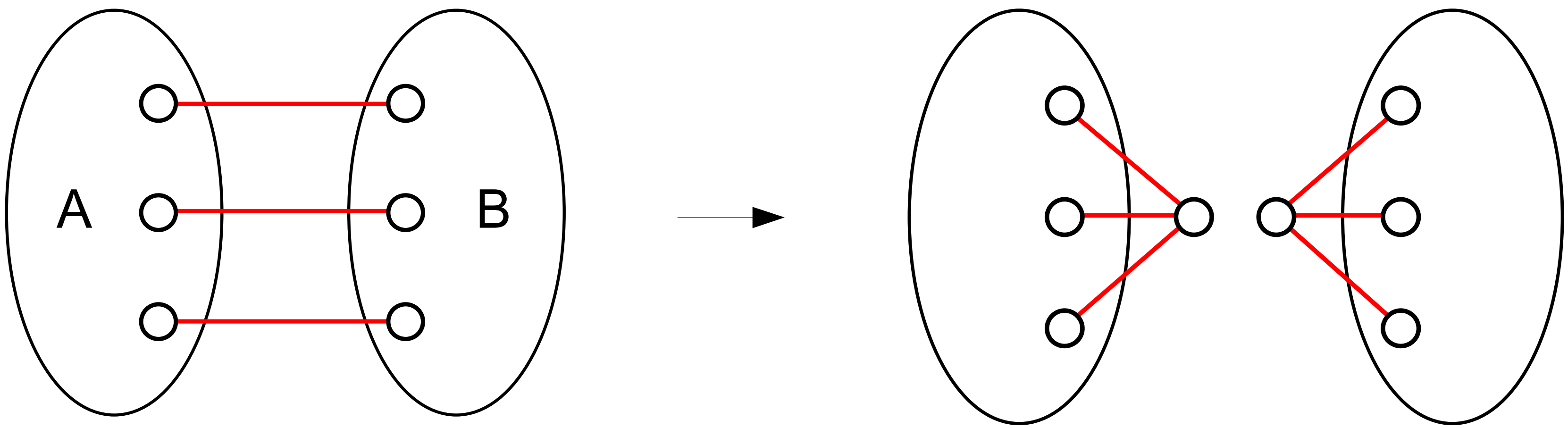

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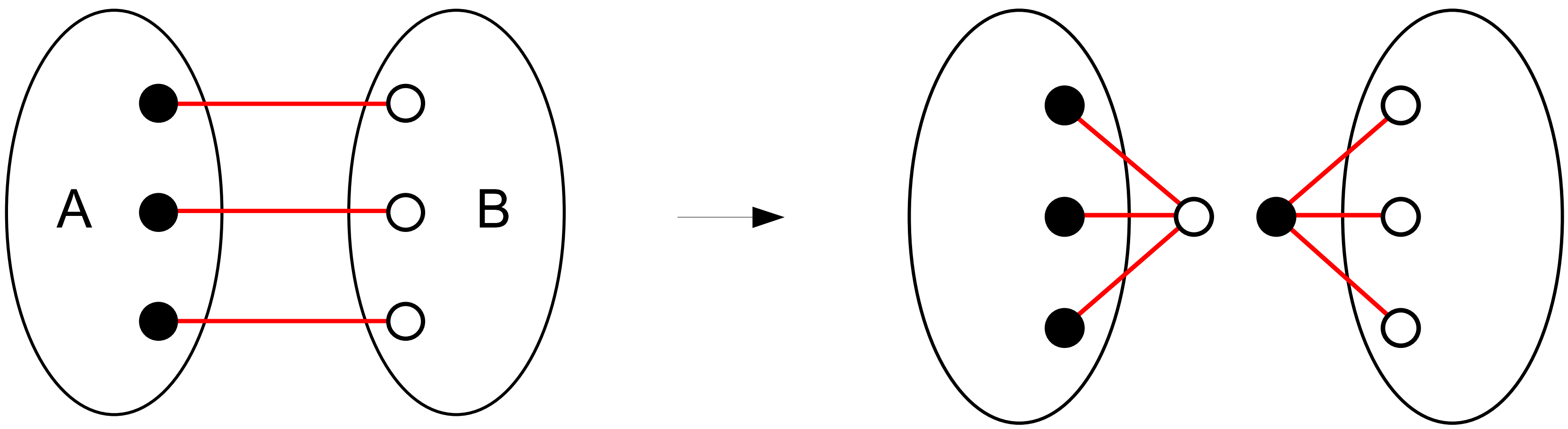

- cubic
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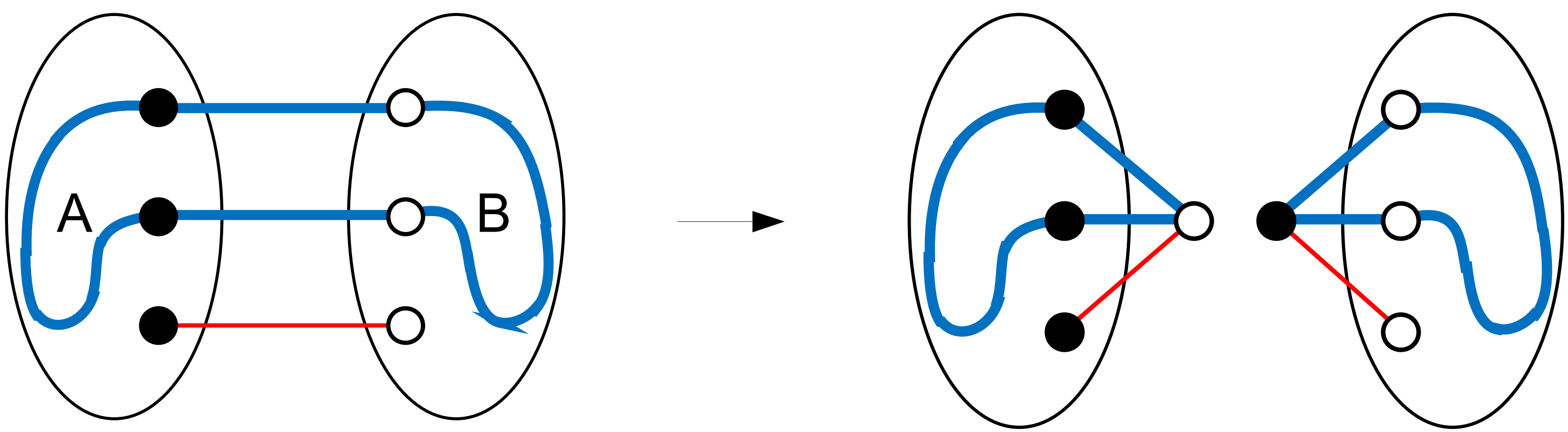

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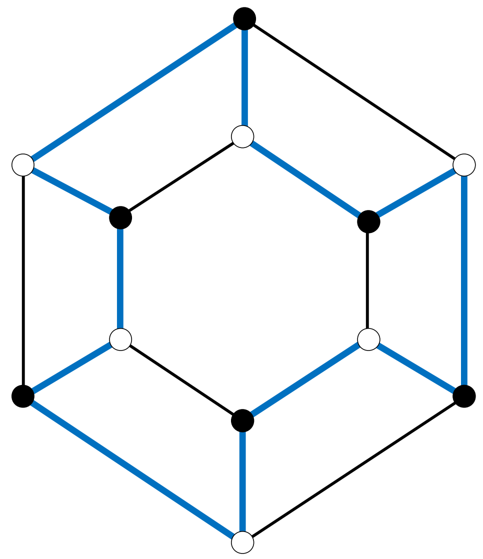
Special Faces

Thm (Goodey 1975). Every Barnette graph that contains only faces of degree 4 and 6 (and at most one special face of degree 8) is Hamiltonian.



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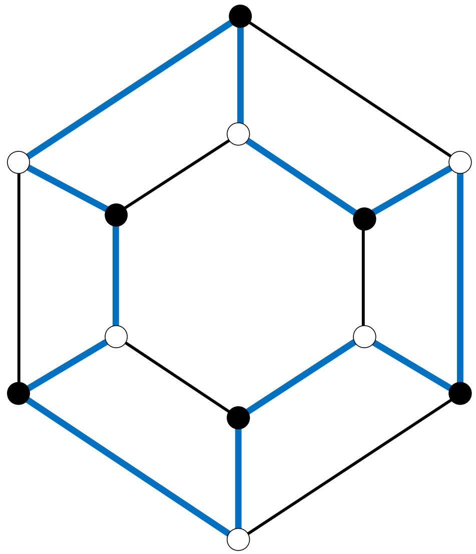

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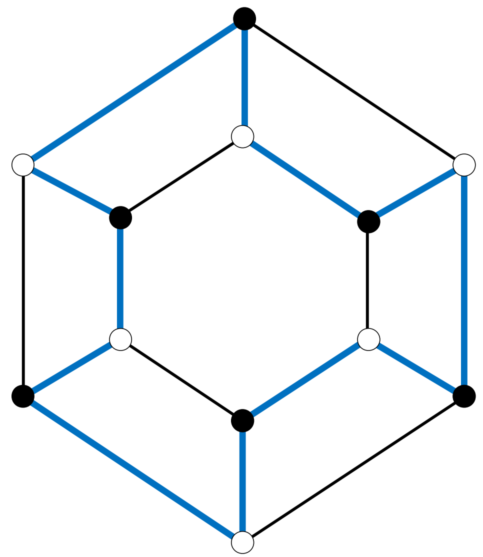

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Computing a Hamiltonian cycle for every such graph is in P.

End of the first talk. Don't run away.