Random discrete structures (MPI, 2014) **Lecturers:** Kunal Dutta and Arijit Ghosh
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By *random variables* or *discrete random variables* we mean random variables taking either finitely many values or countably infinite values.

1. Show that $\ln n \leq H_n \leq \ln n + 1$, where $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

2. (a) For any two event E_1 and E_2, prove

$$\mathbf{Pr}[E_1 \cup E_2] = \mathbf{Pr}[E_1] + \mathbf{Pr}[E_2] - \mathbf{Pr}[E_1 \cap E_2].$$

   (b) Let $E_1, \ldots, E_n$ be any events. Then

$$\begin{aligned}\mathbf{Pr}\left[\bigcup_{i=1}^{n} E_i\right] &= \sum_i \mathbf{Pr}[E_i] - \sum_{i<j} \mathbf{Pr}[E_i \cap E_j] + \sum_{i<j<k} \mathbf{Pr}[E_i \cap E_j \cap E_k] - \ldots \\ &\quad + (-1)^{l+1} \sum_{i_1 < i_2 < \cdots < i_l} \mathbf{Pr}\left[\bigcap_{r=1}^{l} E_{i_r}\right].\end{aligned}$$

   (c) Let $E_1, \ldots, E_n$ be any events, then

$$\mathbf{Pr}\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_i \mathbf{Pr}[E_i].$$

3. (Law of total probability) Let $E_1, \ldots, E_n$ be mutually disjoint events in the probability space Ω such that $\Omega = \bigcup_{i=1}^{n} E_i$. Then

$$\mathbf{Pr}[B] = \sum_{i=1}^{n} \mathbf{Pr}[B \cap E_i] = \sum_{i=1}^{n} \mathbf{Pr}[B \mid E_i]\mathbf{Pr}[E_i].$$

   (Hint: Use 2 (b) and the following result $\mathbf{Pr}[A \cap B] = \mathbf{Pr}[A \mid B]\mathbf{Pr}[B]$)

4. (Bayes' law) Let $E_1, \ldots, E_n$ be mutually disjoint events in the probability space Ω such that $\Omega = \bigcup_{i=1}^{n} E_i$. Then

$$\mathbf{Pr}[E_j \mid B] = \frac{\mathbf{Pr}[E_j \cap B]}{\mathbf{Pr}[B]} = \frac{\mathbf{Pr}[B \mid E_j]\mathbf{Pr}[E_j]}{\sum_{i=1}^{n} \mathbf{Pr}[B \mid E_i]\mathbf{Pr}[E_i]}.$$

5. (a) For any finite collection of discrete random variables $X_1, \ldots, X_n$ with finite expectations

$$\mathbb{E}\left[\sum_{i=1}^{n} X_n\right] = \sum_{i=1}^{n} \mathbb{E}[X_i].$$

   (b) For any constant c and a discrete random variables X, $\mathbb{E}[cX] = c\mathbb{E}[X]$.

   (c) For any random variables X and Y, $\mathbb{E}[X] = \mathbb{E}\left[\exp\left[X \mid Y\right]\right]$.

6. (a) Show that for any convex function $f : \mathbb{R} \to \mathbb{R}$, and any $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

(b) (Jensen's inequality) Let X be a random variable that takes only finitely many values, then show that

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

Prove the above result using 6 (a), and don't assume differentiability of f.

7. If X is a random variable taking only non-negative integer values then,

   (a) $\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbf{Pr}[X \geq i]$.

   (b) $\mathbb{E}[X] \geq \mathbf{Pr}[X > 0]$.

8. (Binomial random variables) Let X be a *binomial random variable* with parameters $n \in 1, 2, \ldots$ and $p \in [0, 1]$, i.e. $\forall j \in \{1, \ldots, n\}$, $\mathbf{Pr}[X = j] = \binom{n}{j} p^j (1-p)^{n-j}$. Then show that $\mathbb{E}[X] = np$.

9. (Geometric random variable) A geometric random variable X with parameter $p \in [0, 1]$ is defined by the following probability distribution: $\forall n \in 1, 2, \ldots, \ldots$

$$\mathbf{Pr}[X = n] = (1-p)^{n-1} p.$$

   Show

   (a) $\mathbb{E}[X] = \frac{1}{p}$.

   (b) (Memorylessness property) $\mathbf{Pr}[X = n + k \mid X > k] = \mathbf{Pr}[X = n]$.

10. (Coupon collector's problem) Suppose each box of cereals contains one of n different coupons independently and uniformly at random from the n possible coupons. Let X be the random variable denoting the number of boxes one needs to buy until they have one coupon of each type. Show that $\mathbb{E}[X] = nH_n$. (Hint: use 5 (a) and 9 (a))

11. (Balancing vectors problem) Let $v_1, \ldots, v_n$ be n vectors in $\mathbb{R}^n$ with $\|v_i\| \leq 1$. Let $p_1, \ldots, p_n \in [0, 1]$ and $w = \sum_{i=1}^{n} p_i v_i$. Then there exits exist $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ so that,

$$\|w - \sum_{i=1}^{n} \epsilon_i v_i\| \leq \frac{\sqrt{n}}{2}.$$

   (Hint: use *expectation*)

12. (Independent set) Let $G = (V, E)$ be a graph with n-vertices and $nd/2$ edges. Show that there exists an independent set of size $\geq n/2d$ in G. (Hint: use *alterations*)

13. (Dominating set) $G = (V, E)$ be a graph with n-vertices and all the vertices have degree $\geq \delta$. Then show that there exists a dominating set of size $\leq \frac{n(1+\ln(\delta+1))}{\delta+1}$ in G. (Hint: use *alterations*)