Minimum-Length Homotopy Basis with a Given Basepoint

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1 Introduction

Let M be an orientable manifold of genus g without a boundary. We fix basepoint $b \in M$ and want to compute the system of loops that generates $\pi_1(M, b)$ (fundamental group of M). This set of generators has $2g$ elements. We want it to be shortest in the following sense: we assume that there is a graph G cellularly embedded into M and we only consider paths that intersect edges of G 'nicely', e.g., paths do not contain vertices and number of intersections with each edge is finite. If we assign weight to each edge of G, we define length of a path as sum of weights of the edges it intersects.

To compute this minimum-length basis we use the greedy approach which works because we actually have an underlying matroid structure.

2 Matroids

Let E be some finite set. We refer to it as *ground set*. Set $I \subset 2^E$ (or, more precisely, pair (E, I))is called an *independent system*, if

1. $\emptyset \in I$
2. $X \subset Y$ and $Y \in I$ implies $X \in I$

An independent system I is called a *matroid*, if

3. If X and Y are elements of I and $|X| < |Y|$, then there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in I$.

Examples of matroids are set of all linearly independent subsets of a finite linear space and edges of forests in some undirected graph.

Maximal elements of I with respect to inclusion are called bases in (E, I).

Suppose now that the ground set E has a positive weighting function $w : E \to \mathbb{R}^+$.

The greedy approach to compute the minimum-weight base B in some independent system (E, I):

Initialize $B := \emptyset$
while $\exists e \in E : B \cup \{e\} \in I$ **do**
    Choose e of the minimal weight and put $B := B \cup \{e\}$, $E = E \setminus \{e\}$

Theorem 2.1. *Independent system (E, I) with positive weight function w is a matroid if and only if the greedy algorithm correctly computes the minimal basis.*

Note: system of loops is *not* a matroid.

3 Homology

Let H denote $H_1(M, \mathbb{Z}2\mathbb{Z})$. If e is an edge of G not in cut locus, then $\sigma(e)$ denotes the loop that we get by adding e to the two shortest paths from vertices of e to the base-point.

Lemma 3.1. *Let L be a set of loops satisfying 3-path condition. There exists a shortest loop l not in L that crosses the cut locus at most once.*

Every system of loops is a homotopy basis.

Lemma 3.2. *Let A be a subset of the cut locus. A disconnects C iff $\sigma(A)$ disconnects M. Here $\sigma(A) = \bigcup_{e \in A} \sigma(e)$.*

Lemma 3.3. *There exists a shortest homology basis made of loops of the form $\sigma(e)$.*

Note that the length of $s \in H$ is defined as

$$|s| = \min\{length(l) | l \text{ is a loop and } l \in s\}$$

It is important that mininum is taken only over loops that pass through the base-point.

Lemma 3.4. *Let L be a set of simple disjoint loops. L disconnects M iff $[L]$ is linearly dependent.*

Theorem 3.5. *A shortest homology basis L of $2g$ loops can be constructed in $O(gn + n \log n)$ time.*

Proposition 3.6. *L from the previous theorem is a shortest system of loops.*

If L generates $\pi_1(M, b)$, it is also a basis of $H_1(M, \mathbb{Z}_2)$ and for each loop ℓ in homology group we have

$$\ell = \sum_{i \in I_\ell} \gamma_i, \qquad \gamma_i \in L$$

γ_i have form $\sigma(e_i)$ and are called greedy factors.

Lemma 3.7. *If γ is a greedy factor of an arbitrary loop ℓ, then $|\gamma| \leq |\ell|$.*

Lemma 3.8. *Let $\{\alpha_1, \ldots, \alpha_{2g}\}$ be a homotopy basis. There exists a permutation $\pi \in S_{2g}$ such that $\gamma_{\pi(i)}$ is a greedy factor of α_i for each i.*

These two lemmas are necessary to prove the following theorem:

Theorem 3.9. *For any 2-manifold M and any basepoint $b \in M$, the greedy homotopy basis is the shortest set of generators of $\pi_1(M, b)$.*