Optimal First Homology Basis on Surfaces

Presented by Michael Kerber
(Scribe: Tiezhi Wang)

1. Problem statement

Goal: given a connected, compact, orientable, combinatorial 2-manifold *M* without boundary, with combinatorial complexity *n* and genus *g*, compute the minimal basis of first homology group $H_1(M)$.

Idea: identify a set *S* of candidate cycles that contains a minimal homology basis, then use the Metro theorem to construct the minimal homology basis.
Time complexity: $O(n^2 \log n + n^2 + ng^3)$.

2. Background and notations

Cycle is sum of closed paths on surface. Cycle is $\mathrm{Z}_2 -$ vector space. *Boundary* is 2-dimentional patch. Boundary is cycle. Cycles C_1 and C_2 are *homologous* if C_1=$\mathrm{C}_2 +$ boundary. Homology group $H_1(M) := \{[c] | c\ is\ cycle\} \cong Z_2^{2g}$, here [c]:=$\{c + b | b\ is\ boundary\}$. $c_{1,} \cdots, c_{2g}$is *homology basis* if $\langle [c_1], \cdots, [c_{2g}] \rangle = H_1(M)$.
$c_{1,} \cdots, c_{2g}$is *minimal* homology basis if $\sum_{i=1}^{2g} length(c_i)$ is minimal among all homology basis. A cycle is *tight* if for any vertices *v*, *w* on c, c contains a shortest path from *v* to *w*.

Lemma 1: Every cycle in a minimal homology basis is tight.
C_x is cut locus with base point *x*, e is an edge of C_x, $\pi(e, x)$ is a cycle combining the path on shortest paths tree from *x* to an vertex adjacent to e, the edge crossing e and the path on shortest paths tree from another vertex adjacent to e back to *x*.
Lemma 2: Every tight cycle is a $\pi(e, x)$ for some base point *x* on the cycle.
Z_x is set of canonical cycles with the form $\pi(\ , x)$. $\bigcup_{\mathrm{x} \in \mathrm{V}} \mathrm{Z}_\mathrm{x}$ contains a minimal homology basis. G_x is the minimal set of loops for base point *x*.
Lemma 3: $\bigcup_{\mathrm{x} \in \mathrm{V}} \mathrm{G}_\mathrm{x}$ contains a minimal homology basis.

Proof: Let G be the greedy set for $\bigcup_{x\in V} Z_x$. G is a minimal homology basis, G=$\{b_1, \cdots b_{2g}\}$, $b_i \in Z_{x_0}$ for some x_0. Because the property of the algorithm, b_i is not generated by all shorter cycles in $\bigcup_{x\in V} Z_{x_0}$, b_i is a part of G_{x_0}.

3. Algorithm

(1) For any $x\in V$, compute G_x.

(2) Sort $\bigcup_{x\in V} G_x$ with regard to length, get a sequence $G_1, \cdots, G_m$, where m$\in O(gn)$.

(3) For i=1,$\cdots, m$

if $[C_i]$ is not genetated by $\langle [C_{1i}], \cdots, [C_{i-1}] \rangle$

add C_i to basis.

4. Runtime analysis

4.1 Runtime of step (1) .

The time consumption for computing G_x is O($n \log n + gn$), so the runtime of step (1) is O($n^2 \log n + gn^2$).

4.2 Runtime of step (2)

Since m$\in O(gn)$ and g is at most n, the runtime of step (2) is O(n^2).

4.3 Runtime of step (3)

Every cycle on the surface can be expressed as a linear combination of $b_1, \cdots, b_{2g}$. In other words, every cycle can be expressed as a zero-one vector of length 2g showing whether b_i is in the linear combination or not, e.g. h(C)=$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$. That can be realized as following.

(a) Fix an point $x_0 \in V$. For the cut locus C_{x_0}, compute the shortest system of loops $b_1, \cdots, b_{2g}$, which is a homology basis.

(b) Express $\pi(e, x)$ as linear combination of $b_1, \cdots, b_{2g}$ by the following way. Label leaves attached to $b_1, \cdots, b_{2g}$ with corresponding numbers. Split spanning tree into two parts corresponding to e, move e to an either component. Collect the label of encountered leaf b_i. If the edge e needs to go over a vertex, take the sum of labels on all braches.

With the expression of all $\pi(e, x)$, any cycle C=$e_1 \cdots e_k$ on surface can be expressed as h(C)=$\sum_1^k h(\pi(e, x_0))$. Then Every homology class can be expressed as a matrix with *2g* rows and *ng* columns, where the basis can be found.

It can be decided that whether $[C_i]$ is generated by $\langle [C_{1i}], \cdots, [C_{i-1}] \rangle$ in $O(g^2)$ time per candidate, and there are *ng* candidates, so the runtime of step (3) is $O(ng^3)$.

4.4 Overall runtime

The overall runtime is $O(n^2 \log n + n^2 + ng^3)$.

5 Remark

The restriction to coefficient fields is necessary. For coefficient rings without division, homology groups are not vector spaces, and thus a maximal linearly independent set of homology classes is not necessarily a basis. [1]

Reference

[1] Jeff Erickson: Combinatorial Optimization of Cycles and Bases. 2000 Mathematics Subject Classification, 2011, Proceedings of Symposia in Applied Mathematics