Shortest Essential Cycles.

(according to Tiezhi Wang Presentation)

Summary.

A loop or cycle is *contractible* if it can be continuously deformed into a point.
On a disk or a sphere, every loop or cycle is *contractible*.
Contractibe ⇒separating.
Non-separating ⇒non-contractible.
Contractibe ⇔ Zero in the homotopy group.
Separating ⇔ Zero in the $\mathbb{Z}_2$-homology group.
Lemma1. A simple loop (or cycle) is *contractible* if and only if it bounds a disk.
Given an edge $e \in C$, the loop $\sigma(e)$ is defined as a loop with basepoint b that follows the shortest path tree to go from its root b to a face incident with e, crosses e, and goes back from the other face incident with e to the root. This can be done so that all the loops $\sigma(e)$ are simple and disjoint. (except, of course, at their basepoint b we shall omit this triviality in the sequel).
Define the weight of an edge e of C to be the length of the corresponding loop $\sigma(e)$.
Lemma 2. A simple loop (or cycle) is contractible if and only if it bounds a disk.
A set L of loops satisfies the 3-path condition if, for any point $a \neq b$ and any three paths p, q, and r from b to a, if p.$\bar{q}$ and q.$\bar{r}$ belong to L, then p.$\bar{r}$ belongs to L.
Lemma 3. The set of contractible loops satisfies the 3-path condition.
Cut locus. Let T be the shortest path tree from b to a point in each face of G^* .Remove all edges of G^*crossed by T.
Lemma 4. Let L be a set of loops satisfying the 3-path condition. Some shortest loop not in L crosses the cut locus C at most once.
Lemma 5. Some shortest non-contractible loop has the form $\sigma(e)$.
Lemma 6. Let e be an edge of C. Then $\sigma(e)$ is contractible if and only if some component of C\e is a tree.
Theorem 1. Finding a shortest non-contractible loop can be done in O(n log n) time. The loop computed is simple.
Corollary 1.Finding a shortest non-contractible cycle can be done in O(n^2 log n) time.

- Choose a vertex as basepoint.
- Compute C, assign every e of C a weight that is length of σ(e). O(nlogn)
- Eliminate the edges e such that at least one component of C\e is a tree. O(n)
- Select the e with minimum weight in remaining edges of C. O(n)
  Overall O(n^2 log n)

A simple loop ℓ is separating if $\varphi\backslash\backslash\ell$ is not connected. A simple contractible loop bounds a disk, hence is separating; the converse is false.
A chain $\acute{E} \in E$ is a homology cycle if every vertex of V is incident with an even number of edges of $\acute{E}$.
A chain $\acute{E} \in E$ is a homology boundary if the faces of G can be colored black and white so that $\acute{E}$ is the set of edges of E with exactly one black and one white incident face.
Lemma 7. A simple loop ℓ in G disconnects φ if and only if it is a homology boundary.
So the notion of homology boundary extends the notion of being separating.