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May 8, 2015
Deadline: lecture on May 18, 2015

Algorithmic Game Theory

Summer 2015

Exercise Set 2

Exercise 1: (4 Points)
Give an example of a *symmetric network congestion game* (strategies are s-t paths in a graph with the same s and t for all players) with non-decreasing delay functions in which there are at least two pure Nash equilibria with different social costs.
The social cost is defined to be the sum of delays of all players $\sum_{i \in \mathcal{N}} c_i(S)$.

Exercise 2: (5 + 1 Points)
In a *weighted congestion games*, each player $i \in \mathcal{N}$ has an individual weight $w_i > 0$. The delay on depends on the sum of the weights of the players choosing this resource rather than just on their number. Formally, we could replace the definition of $n_r(S)$ by $n_r(S) = \sum_{r \in S_i} w_i$.

(a) Show that weighted congestion games do not fulfill the finite-improvement property, even with two players, three resources, and two strategies per player.
Hint: Consider $\mathcal{N} = \{1, 2\}$, $w_1 = 1$, $w_2 = 2$, $\mathcal{R} = \{a, b, c\}$, $\Sigma_1 = \{\{a\}, \{b, c\}\}$, $\Sigma_2 = \{\{b\}, \{a, c\}\}$. Choose delay functions such that

$$(\{a\}, \{b\}) \to (\{a\}, \{a, c\}) \to (\{b, c\}, \{a, c\}) \to (\{b, c\}, \{b\}) \to (\{a\}, \{b\})$$

is an improvement sequence.

(b) Why does a solution to (a) imply that there is even no pure Nash equilibrium?

Exercise 3: (4 + 6 Points)
Load-balancing games are a special case of weighted congestion games. There are m machines of identical speeds. Player i is in charge of one job of weight $w_i > 0$. Every player may choose a machine to process this job; his strategy set is therefore $[m]$. The costs are given by the time that it takes to process all jobs. That is, player i's cost in strategy profile s is given as

$$c_i(s) = \sum_{i' : s_{i'} = s_i} w_{i'} \quad .$$

(Note that this sum always includes w_i.)

(a) Show that there is a strategy profile s that is a pure Nash equilibrium and minimizes $\max_{i \in \mathcal{N}} c_i(s)$ at the same time.

(b) Show that load-balancing games fulfill the finite-improvement property.
Hint: Consider the vector of player costs, ordered non-increasingly, and how this vector changes with respect to the lexicographical order.