Algorithmic Game Theory, Summer 2015 — Lecture 1 (5 pages)

Game-Theory Basics

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Example 1.1 (Prisoner's Dilemma)**.** *Two criminals are interrogated separately. Each of them has two possible strategies: (C)onfess, remain (S)ilent. Confessing yields a smaller sentence if the other one is silent. If both confess, the sentence is larger for both (4 years) compared to when they both remain silent (2 years).*

	S	C
S	2 2	1 5
C	5 1	4 4

- *If both players remain (S)ilent, the total cost is smallest.*
- *If both players (C)onfess, the cost is larger for both of them.*
- *Still, for each player confessing is always the preference!*

Definition 1.2. *A* (normal form, cost minimization) game *is a triple* $(\mathcal{N}, (S_i)_{i \in N}, (c_i)_{i \in N})$ *where*

- $\mathcal{N}$ *is the set of players,* $n = |\mathcal{N}|$*,*
- S_i *is the set of* (pure) strategies *of player* i*,*
- $S = \prod_{i \in \mathcal{N}} S_i$ *is the set of states,*
- $c_i \colon S \to \mathbb{R}$ *is the cost function of player* $i \in \mathcal{N}$*. In state* $s \in S$*, player* i *has a cost of* $c_i(s)$*.*

We denote by $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ a state s without the strategy s_i. This notation allows us to concisely define a unilateral deviation of a player. For $i \in \mathcal{N}$, let $s \in S$ and $s'_i \in S_i$, then $(s'_i, s_{-i}) = (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$.

Definition 1.3. *A pure strategy* s_i *is called a* dominant strategy *for player* $i \in \mathcal{N}$ *if* $c_i(s_i, s_{-i}) \leq c_i(s'_i, s_{-i})$ *for every* $s'_i \in S_i$ *and every* s_{-i}*.*

Definition 1.4. *A state* $s \in S$ *is called a* dominant strategy equilibrium *if for every player* $i \in \mathcal{N}$ *strategy* $s_i \in S_i$ *is a dominant strategy.*

Not every game has a dominant strategy equilibrium.

Example 1.5 (Battle of the Sexes)**.** *Suppose Angelina and Brad go to the movies. Angelina prefers watching movie A, Brad prefers watching movie B. However, both prefer watching a movie together to watching movies separately.*

	A	B
A	1 2	6 6
B	5 5	2 1

There is no dominant strategy of either of the two player: In state (A,A) the preference for both is A. In state (B,B) the preference for both is B.

What is a likely outcome in this situation?

Definition 1.6. *A strategy s_i is called a* best response *for player $i \in \mathcal{N}$ against a collection of strategies s_{-i} if $c_i(s_i, s_{-i}) \leq c_i(s_i', s_{-i})$ for all $s_i' \in S_i$.*

Note: s_i dominant strategy if and only if s_i best response for all s_{-i}.

Definition 1.7. *A state $s \in S$ is called a* pure Nash equilibrium *if s_i is a best response against the other strategies s_{-i} for every player $i \in \mathcal{N}$.*

So, a pure Nash equilibrium is stable against unilateral deviation. No player can reduce his cost by only changing his only strategy.

Not every game has a pure Nash equilibrium.

Example 1.8 (Rock-Paper-Scissors)**.** *The well-known game rock-paper-scissors can be represented by the following cost matrix.*

	R	*P*	*S*
R	*0* / *0*	*-1* / *1*	*1* / *-1*
P	*1* / *-1*	*0* / *0*	*-1* / *1*
S	*-1* / *1*	*1* / *-1*	*0* / *0*

There is no pure Nash equilibrium: In each of the nine states, at least one of the two players does not play a best response.

Definition 1.9. *A* mixed strategy *σ_i for player i is a probability distribution over the set of pure strategies S_i.*

We will only consider the case of finitely many pure strategies and finitely many players. In this case, we can write a mixed strategy σ_i as $(\sigma_{i,s_i})_{s_i \in S_i}$ with $\sum_{s_i \in S_i} \sigma_{i,s_i}$. The cost of a mixed state σ for player i is

$$c_i(\sigma) = \sum_{s \in S} p(s) \cdot c_i(s) \ ,$$

where $p(s) = \prod_{i \in \mathcal{N}} \sigma_{i,s_i}$ is the probability that the outcome is pure state s.

Definition 1.10. *A mixed strategy σ_i is a* (mixed) best-response strategy against a collection of mixed strategies σ_{-i} *if $c(\sigma_i, \sigma_{-i}) \leq c_i(\sigma_i', \sigma_{-i})$ for all other mixed strategies σ_i'.*

Definition 1.11. *A mixed state σ is called a* mixed Nash equilibrium *if σ_i is a best-response strategy against σ_{-i} for every player $i \in \mathcal{N}$.*

Note that every pure strategy is also a mixed strategy and every pure Nash equilibrium is also a mixed Nash equilibrium.

It is enough to only consider deviations to pure strategies.

Lemma 1.12. *A mixed strategy σ_i is a best-response strategy against σ_{-i} if and only if $c_i(\sigma_i, \sigma_{-i}) \leq c_i(s_i', \sigma_{-i})$ for all pure strategies $s_i' \in S_i$.*

Proof. The "only if" part is trivial: Every pure strategy is also a mixed strategy.

For the "if" part, let σ_{-i} be an arbitrary mixed strategy profile for all players except for i. Furthermore, let σ_i be a mixed strategy for player i such that $c_i(\sigma_i, \sigma_{-i}) \leq c_i(s_i', \sigma_{-i})$ for all pure strategies $s_i' \in S_i$.

Observe that for any mixed strategy σ_i', we have $c_i(\sigma_i', \sigma_{-i}) = \sum_{s_i' \in S_i} \sigma_{i,s_i'} c_i(s_i', \sigma_{-i}) \geq \min_{s_i' \in S_i} c_i(s_i', \sigma_{-i})$. Using $\min_{s_i' \in S_i} c_i(s_i', \sigma_{-i}) \geq c_i(\sigma_i, \sigma_{-i})$, we are done. □

While dominant-strategy and pure Nash equilibria do not necessarily exist, mixed Nash equilibria always exist if the number of players and the number of strategies is finite.

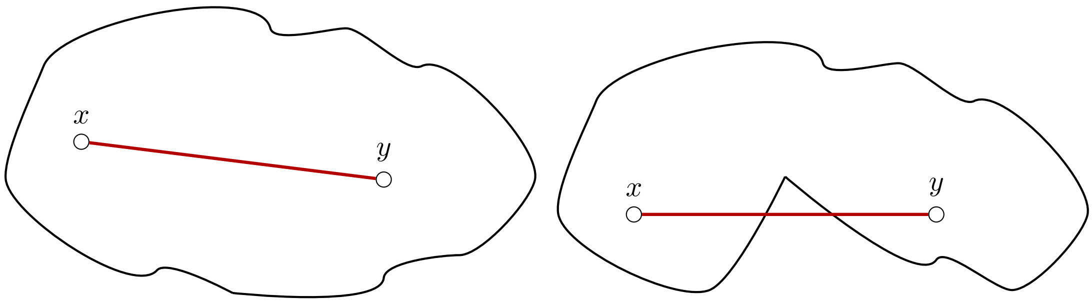
Theorem 1.13 (Nash Theorem)**.** *Every finite normal form game has a mixed Nash equilibrium.*

We will use Brouwer's fixed point theorem to prove it.

Theorem 1.14 (Brouwer Fixed Point Theorem)**.** *Every continuous function $f\colon D \to D$ mapping a compact and convex nonempty subset $D \subseteq \mathbb{R}^m$ to itself has a fixed point $x^* \in D$ with $f(x^*) = x^*$.*

As a reminder, these are the definitions of the terms used in Brouwer's fixed point theorem. Here, $\|\cdot\|$ denotes an arbitrary norm, for example, $\|x\| = \max_i |x_i|$.

- A set $D \subseteq \mathbb{R}^m$ is *convex* if for any $x, y \in D$ and any $\lambda \in [0,1]$ we have $\lambda x + (1-\lambda) y \in D$.



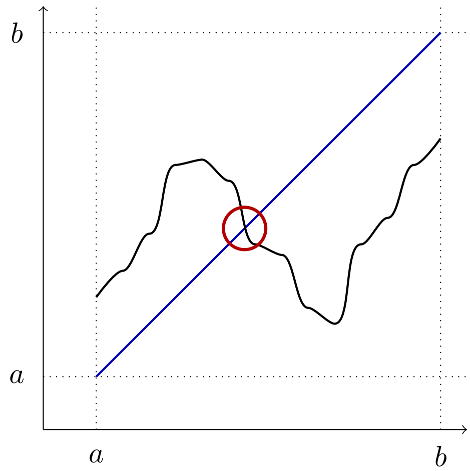
convex not convex

- A set $D \subseteq \mathbb{R}^m$ is *compact* if and only if it is closed and bounded.
- A set $D \subseteq \mathbb{R}^m$ is *bounded* if and only if there is some bound $r \geq 0$ such that $\|x\| \leq r$ for all $x \in D$.
- A set $D \subseteq \mathbb{R}^m$ is *closed* if it contains all its limit points. That is, consider any convergent sequence $(x_n)_{n\in\mathbb{N}}$ within D, i.e., $\lim_{n\to\infty} x_n$ exists and $x_n \in D$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n \in D$.

$$[0,1] \text{ is closed and bounded}$$
$$(0,1] \text{ is not closed but bounded}$$
$$[0,\infty) \text{ is closed and unbounded}$$

- A function $f\colon D \to \mathbb{R}^m$ is *continuous at a point* $x \in D$ if for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $y \in D$: If $\|x - y\| < \delta$ then $\|f(x) - f(y)\| < \epsilon$.
  f is called *continuous* if it is continuous at every point $x \in D$.

Equivalent formulation of Brouwer's fixed point theorem in one dimension:
For all $a, b \in \mathbb{R}$, $a < b$, every continuous function $f\colon [a,b] \to [a,b]$ has a fixed point.



Proof of Theorem 1.13. Consider a finite normal form game. Without loss of generality let $\mathcal{N} = \{1, \dots, n\}$, $S_i = \{1, \dots, m_i\}$. So the set of mixed states X can be considered a subset of $\mathbb{R}^m$ with $m = \sum_{i=1}^n m_i$.

Exercise: Show that X is convex and compact.

We will define a function $f\colon X \to X$ that transforms a mixed strategy profile into another mixed strategy profile. The fixed points of f are shown to be the mixed Nash equilibria of the game.

For mixed state x and for $i \in \mathcal{N}$ and $j \in S_i$, let

$$\phi_{i,j}(x) = \max\{0, c_i(x) - c_i(j, x_{-i})\} \ .$$

So, $\phi_{i,j}(x)$ is the amount by which player i's cost would reduce when unilaterally moving from x to j if this quantity is positive, otherwise it is 0.

Observe that by Lemma 1.12 a mixed state x is a Nash equilibrium if and only if $\phi_{i,j}(x) = 0$ for all $i = 1, \dots, n$, $j = 1, \dots, m_i$.

Define $f\colon X \to X$ with $f(x) = x' = (x'_{1,1}, ..., x'_{n,m_n})$ by

$$x'_{i,j} = \frac{x_{i,j} + \phi_{i,j}(x)}{1 + \sum_{k=1}^{m_i} \phi_{i,k}(x)}$$

for all $i = 1, \dots, n$ and $j = 1, \dots, m_i$.

Observe that $x' \in X$. That means, $f\colon X \to X$ is well defined. Furthermore, f is continuous. Therefore, by Theorem 1.14, f has a fixed point, i.e., there is a point $x^* \in X$ such that $f(x^*) = x^*$.

We only need to show that every fixed point x^* of f is a mixed Nash equilibrium. So, in other words, we need to show that $f(x^*) = x^*$ implies that $\phi_{i,j}(x^*) = 0$ for all $i = 1, \dots, n$, $j = 1, \dots, m_i$.

Fix some $i \in \mathcal{N}$. Once we have shown that $\phi_{i,j}(x^*) = 0$ for $j = 1, \dots, m_i$, we are done. We observe that there is j' with $x^*_{i,j'} > 0$ and $c_i(x^*) \leq c_i(j', x^*_{-i})$ because $c_i(x^*)$ is defined to be $\sum_{j=1}^{m_i} x^*_{i,j} \cdot c_i(j, x^*_{-i})$. So, it is the weighted average of all costs and it is impossible that every pure strategy has strictly smaller cost then the weighted average. For this j', $\phi_{i,j'}(x^*) = \max\{0, c_i(x^*) - c_i(j, x^*_{-i})\} = 0$.

We now use the fact that x^* is a fixed point. Therefore, we have

$$x^*_{i,j'} = \frac{x^*_{i,j'} + \phi_{i,j'}(x^*)}{1 + \sum_{k=1}^{m_i} \phi_{i,k}(x^*)} = \frac{x^*_{i,j'}}{1 + \sum_{k=1}^{m_i} \phi_{i,k}(x^*)} \ .$$

As $x^*_{i,j'} > 0$, we also have

$$1 = \frac{1}{1 + \sum_{k=1}^{m_i} \phi_{i,k}(x^*)} \enspace ,$$

and so

$$\sum_{k=1}^{m_i} \phi_{i,k}(x^*) = 0 \enspace .$$

Since $\phi_{i,k}(x^*) \geq 0$ for all k, we have to have $\phi_{i,k}(x^*) = 0$ for all k. This completes the proof. □