Algorithmic Game Theory, Summer 2015	Lecture 13 (3 pages)

Mechanism Design without Money

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Due to the impossibility result by Gibbard and Satterthwaite, we know that on the general domain of preferences, one cannot implement any non-dictatorial social choice functions in dominant strategies. In the previous few lectures, we looked at a particular class of domains, namely the quasilinear environment, and showed that there exists interesting incentive compatible mechanisms. Quasilinear environment models the situation involving the transfer of money. Nevertheless, there are a number of situations where monetary compensation is infeasible or illegal. Typical examples include voting, organ donation, school allocation, etc. In this lecture we briefly talk about one of such applications.

1 House Allocation Problem

The following house allocation problem is proposed by Shapley and Scarf: assuming that there is a set N of n agents. Each agent owns one house initially and has a total-order preference $\succ_i$ over all n houses. The goal is to reallocate the houses among the agents to make them better off.

Because the preferences are private information held by each individual agent, firstly we would of course want to design a reallocation mechanism in which no agent has incentive to misreport their preferences. But only having this property is not impressive enough - the mechanism that let each agent keeps his initially house is incentive compatible. Hence our goal is to design some incentive compatible mechanism that can also certain level of optimality. In this problem, an unique feature is that the agents controls all the resources and any subset of agents may therefore subverted the mechanism and trade among themselves. Thus we would like design a mechanism that can prevent agents from opting out. To this end we introduce the notions of *blocking coalitions* and *cores*.

Number each house by the number of the agent who owns the house initially. An allocation of the houses is a n-dimensional vector $\mathbf{a} = (a_1, \dots, a_n)$ in which a_i denotes the number of the house assigned to agent i. We require that $a_i \neq a_j$ for each $i \neq j$. And $a_i = i$ for all i is the initial allocation.

Definition 13.1 (Blocking Coalition)**.** *Given an allocation* $\mathbf{a}$*, a set* S *of agents is called a* **blocking coalition** *for allocation* $\mathbf{a}$*, if there exists another allocation* $\mathbf{b}$*, such that*

- *for every* $i \in S$*,* $b_i \in S$*, and*
- $b_i \succ_i a_i$ *or* $b_i = a_i$ *for every* $i \in S$*, and for at least one* $j \in S$*,* $b_j \succ_j a_j$*.*

Definition 13.2 (Core)**.** *The set of allocations that does not have any blocking coalition is called the* **core***.*

Intuitively, in any core allocation, no coalition of agents can make all of its members better off (or remain unchanged) via internal reallocations. The concept of core applies for a more general class of so called *cooperative games*. The house allocation problem discussed here is one example of a cooperative game with nontransferable utilities. The interested reader is referred to Chapter 15 (Section 15.2) of the AGT book for more information about the cooperative games.

With this house allocation problem, Gale proposed the following *Top Trading Cycle Algorithm (TTCA)* that can always produce an (unique) core allocation in a truthful fashion.

Top Trading Cycle Algorithm

While agents remain, repeat the follow procedures:

1. Construct a directly graph in which each vertex is an agent. For each agent i, build a direct edge (i, j) if house j is agent i's favorite remaining house. This process will give us a direct graph G in which every vertex has out-degree one.
2. Find all directed cycles of this graph (including self-loops). It is easy to see that G has at least one such cycles, and all cycles are vertex-disjoint.
3. Let S be the set of agents incident to these cycles. Reallocate every agent in S his most-preferred house, as the cycles suggested. Note that during this procedure only agents in S will reallocate houses among themselves.
4. Remove all agents (and their houses) in S from the game.

Theorem 13.3. *The TTCA induces an incentive compatible mechanism.*

Proof. Let N_j denote the agents incident to the cycles in the jth round of the algorithm. For any agent i, assumes that $i \in N_j$ for some j. This means that house i is not pointed to by any agent in $N_1 \cup N_2 \cup \cdots \cup N_{j-1}$. Because otherwise agent i would be chosen in some earlier round instead of round j. Thus, whatever agent i reports, he will not receive any house owned by an agent in $S = N_1 \cup N_2 \cup \cdots \cup N_{j-1}$. In the meanwhile, by reporting truthfully, he can get his favorite house outside of set S, hence truth-telling is a dominant strategy for agent i. □

Theorem 13.4. *The allocation computed by the TTCA is the unique core allocation for every house allocation problem.*

Proof. First we prove that the allocation computed by the TTCA (denoted by $\mathbf{a}$) is in the core. Assume by contradiction that there is a blocking coalition S for this allocation $\mathbf{a}$. Let $\mathbf{b}$ be the improved allocation for S defined by this blocking coalition. Let j be the smallest value such that there exists an agent $i \in N_j \cap S$ with $b_i \succ_i a_i$. Define N_j as in the proof of the previous theorem. By the algorithm, agent i gets his favorite house a_i outside set $N_1 \cup N_2 \cup \cdots \cup N_{j-1}$. Hence $b_i \succ_i a_i$ implies $b_i \in N_1 \cup N_2 \cup \cdots \cup N_{j-1}$. Let C be the cycle picked by the algorithm that contains b_i. It is easy to check that $C \subseteq S \cap N_{j'}$ for some $j' < j$. Let $i^* \in C$ be the agent who is assigned to house b_i in allocation $\mathbf{a}$. Then in $\mathbf{b}$ he must receive a different and better house. This contradicts the fact that j is the earliest round in which such event happens.

Next We show that the core allocation is unique. In the allocation computed by the TTCA, all agents in N_1 receives their favorite houses. Thus every core allocation must assign agents in N_1 to the houses just as the TTCA assigns them. Because otherwise set N_1 will be a blocking coalition. We can then use this fact to claim the same argument for set N_2, that is, if an allocation is in the core, agents in N_2 must be assigned the same houses that they receive under the TTCA. Repeat the above argument procedure, we have that if an allocation is in the core, it must be the one determined by the TTCA. □

Recommended Literature

- Chapter 10.3 in the AGT book.
- Tim Roughgarden's lecture notes `http://theory.stanford.edu/~tim/f13/l/l9.pdf` Section 3 and lecture video `https://youtu.be/zV6yH3-AdEg`
- L. Shapley and H. Scarf. *On cores and indivisibility.* Journal of Mathematical Economics, 1(1):23-37, 1974