Algorithmic Game Theory, Summer 2015	Lecture 2 (3 pages)

Introduction to Congestion Games

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1 Definitions and Preliminaries

Definition 2.1 (Congestion Game (Rosenthal 1973))**.** *A* congestion game *is a tuple* $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i\in\mathcal{N}}, (d_r)_{r\in\mathcal{R}})$ *with*

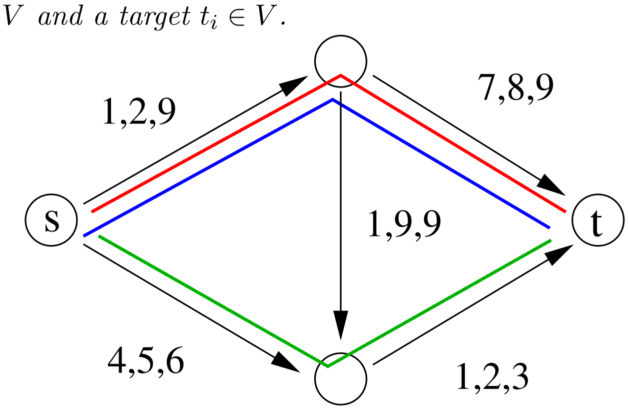
- $\mathcal{N} = \{1, \ldots, n\}$*, set of players*
- $\mathcal{R} = \{1, \ldots, m\}$*, set of resources*
- $\Sigma_i \subseteq 2^{\mathcal{R}}$*, strategy space of player* i
- $d_r \colon \{1, \ldots, n\} \to \mathbb{Z}$*, delay function or resource* r

For any state $S = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \Sigma_n$*,*

- $n_r(S) = |\{i \in \mathcal{N} \mid r \in S_i\}|$*: number of players with* $r \in S_i$
- $d_r(n_r(S))$*: delay of resource* r
- $\delta_i(S) = \sum_{r\in S_i} d_r(n_r)$*: delay of player* i

The cost *of player* i *in state* S *is* $c_i(S) = \delta_i(S)$*, that is, players aim at minimizing their delays.*

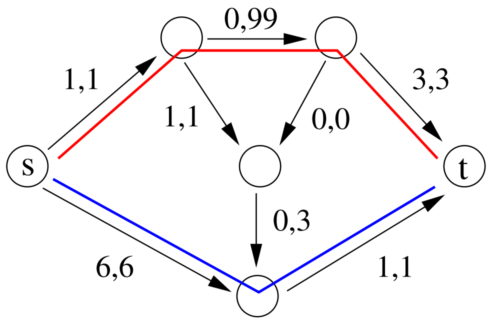
Example 2.2 (Network Congestion Game)**.** *Given a directed graph* $G = (V, E)$ *with delay functions* $d_e \colon \{1, \ldots, n\} \to \mathbb{Z}$*,* $e \in E$*. Player* i *wants to allocate a path of minimal delay between a source* $s_i \in V$ *and a target* $t_i \in V$*.*



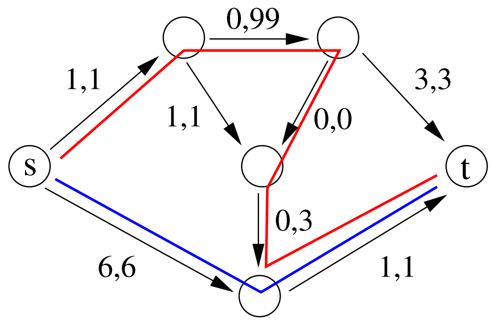
In this example, $\mathcal{N} = \{1, 2, 3\}$*,* $\mathcal{R} = E$*,* Σ_i *= set of s-t paths.*
This game is symmetric*, i.e., all players have the same set of strategies.*

Example 2.3. *A sequence of (best response) improvement steps:*

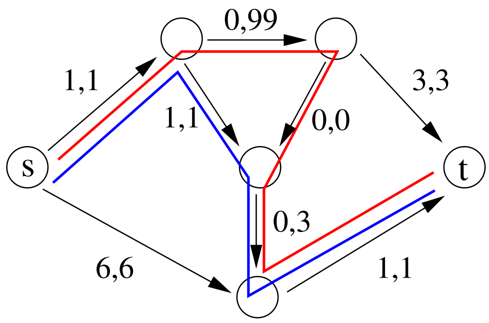
start:



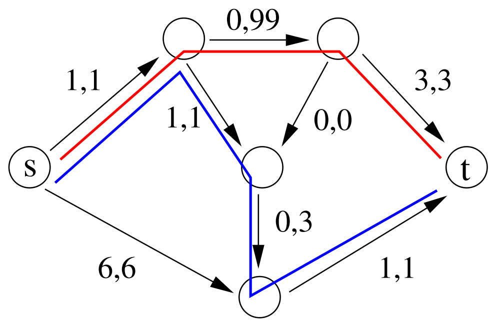
after first improvement (red player):



after second improvement (blue player):



after third improvement (red player):



reached pure nash equilibrium

Questions

- Does every congestion game posses a pure Nash equilibrium?
- Is every sequence of improvement steps finite?
- How many steps are needed to reach a (pure) Nash equilibrium?
- What is the complexity of computing (pure) Nash equilibria in congestion games?

2 Existence of Pure Nash Equilibria

Theorem 2.4 (Rosenthal 1973)**.** *For every congestion game, every sequence of improvement steps is finite.*

This result immediately implies

Corollary 2.5. *Every congestion game has at least one pure Nash equilibrium.*

Proof of Theorem 2.4. Rosenthal's analysis is based on a potential function argument. For every state S, let

$$\Phi(S) = \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} d_r(k) \ .$$

This function is called *Rosenthal's potential function.*

Lemma 2.6. *Let S be any state. Suppose we go from S to a state S' by an improvement step of player i decreasing his delay by $\Delta > 0$. Then $\Phi(S') = \Phi(S) - \Delta$.*

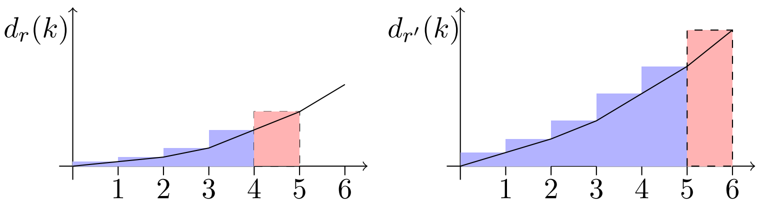


Figure 1: Proof of Lemma 2.6: The contribution of two resources r and r' to the potential is the shaded area. If a player changes from r' to r, his delay changes exactly as the potential value (difference of red areas).

Proof. The potential $\phi(S)$ can be calculated by inserting the players one after the other in any order, and summing the delays of the players at the point of time at their insertion.

Without loss of generality player i is the last player that we insert when calculating $\Phi(S)$. Then the potential accounted for player i corresponds to the delay of player i in state S. When going from S to S', the delay of i decreases by Δ, and, hence, Φ decreases by Δ as well (see Figure 2 for an example. □

The lemma shows that Φ is a so-called *exact potential*, i.e., if a single player decreases its latency by a value of $\Delta > 0$, then Φ decreases by exactly the same amount.

Further observe that

(i) the delay values are integers so that, for every improvement step, $\Delta \geq 1$,

(ii) for every state S, $\Phi(S) \leq \sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$,

(iii) for every state S, $\Phi(S) \geq -\sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$.

Consequently, the number of improvements is upper-bounded by $2 \cdot \sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$ and hence finite. □