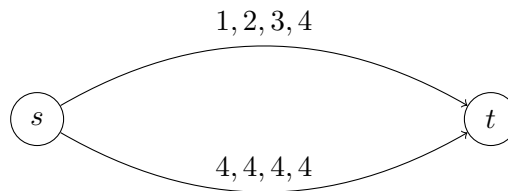


Price of Anarchy

Instructor: Thomas Kesselheim

One of the main goals of algorithmic game theory is to quantify the performance of a system of selfish agents. Usually the “social cost” incurred by all players is higher than if there is a central authority taking charge to minimize social cost. Throughout this lecture, we will assume that the *social cost* is defined to be the sum of all players’ cost; for a state s let $cost(s) = \sum_{i \in \mathcal{N}} c_i(s)$ denote the social cost of s . Sometimes it makes more sense to consider the *maximum* cost incurred by any player.

Consider the following symmetric network congestion game with four players.



There are five kinds of states:

- (a) all players use the top edge, social cost: 16
- (b) three players use the top edge, one player uses the bottom edge, social cost: 13
- (c) two players use the top edge, two players use the bottom edge, social cost: 12
- (d) one player uses the top edge, three players use the bottom edge, social cost: 13
- (e) all players use the bottom edge, social cost: 16

Observe that only states of kind (a) and (b) can be pure Nash equilibria. The social cost, however, is minimized by states of kind (c). Therefore, when considering pure Nash equilibria, due to selfish behavior, we lose up to a factor of $\frac{16}{12}$ and at least a factor of $\frac{13}{12}$. The first quantity is referred to as the price of anarchy, the latter as the price of stability.

The price of anarchy and the price of stability of course depend on what kind of equilibria we consider. To define these two notions formally, we therefore assume that there is a set \mathbf{Eq} of probability distributions over the set of states S , which correspond to equilibria. In the case of pure Nash equilibria, each of these distributions concentrates all its mass on a single point.

Definition 5.1. *Given a cost-minimization game, let \mathbf{Eq} be a set of probability distributions over the set of states S . For some probability distribution p , let $cost(p) = \sum_{s \in S} p(s)cost(s)$ be the expected social cost. The price of anarchy for \mathbf{Eq} is defined as*

$$PoA_{\mathbf{Eq}} = \frac{\max_{p \in \mathbf{Eq}} cost(p)}{\min_{s \in S} cost(s)} .$$

The price of stability for \mathbf{Eq} is defined as

$$PoS_{\mathbf{Eq}} = \frac{\min_{p \in \mathbf{Eq}} cost(p)}{\min_{s \in S} cost(s)} .$$

Given the respective equilibria exist, we have

$$1 \leq PoS_{\text{CCE}} \leq PoS_{\text{MNE}} \leq PoS_{\text{PNE}} \leq PoA_{\text{PNE}} \leq PoA_{\text{MNE}} \leq PoA_{\text{CCE}} .$$

1 Smooth Games

A very helpful technique to derive lower bounds on the price of anarchy is *smoothness*.

Definition 5.2. A game is called (λ, μ) -smooth for $\lambda > 0$ and $\mu \leq 1$ if, for every pair of states $s, s^* \in S$, we have

$$\sum_{i \in \mathcal{N}} c_i(s_i^*, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) .$$

Observe that this condition needs to hold for *all* states $s, s^* \in S$, as opposed to only pure Nash equilibria or only social optima. We consider the cost that each player incurs when unilaterally deviating from s to his strategy in s^* . If the game is smooth, then we can upper-bound the sum of these costs in terms of the social cost of s and s^* .

Smoothness directly gives a bound for the price of anarchy, even for coarse correlated equilibria.

Theorem 5.3. In a (λ, μ) -smooth game, the PoA for coarse correlated equilibria is at most

$$\frac{\lambda}{1 - \mu} .$$

Proof. Let s be distributed according to a coarse correlated equilibrium and s^* be an optimum solution, which minimizes social cost. Then:

$$\begin{aligned} \mathbf{E}[\text{cost}(s)] &= \sum_{i \in \mathcal{N}} \mathbf{E}[c_i(s)] \\ &\leq \sum_{i \in \mathcal{N}} \mathbf{E}[c_i(s_i^*, s_{-i})] && \text{(as } s \text{ is distributed according to CCE)} \\ &= \mathbf{E} \left[\sum_{i \in \mathcal{N}} c_i(s_i^*, s_{-i}) \right] && \text{(by linearity of expectation)} \\ &\leq \mathbf{E}[\lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s)] && \text{(by smoothness)} \end{aligned}$$

On both sides subtract $\mu \cdot \mathbf{E}[\text{cost}(s)]$, this gives

$$(1 - \mu) \cdot \mathbf{E}[\text{cost}(s)] \leq \lambda \cdot \text{cost}(s^*)$$

and rearranging yields

$$\frac{\mathbf{E}[\text{cost}(s)]}{\text{cost}(s^*)} \leq \frac{\lambda}{1 - \mu} .$$

□

That is, in a (λ, μ) -smooth game, we have

$$PoA_{\text{PNE}} \leq PoA_{\text{MNE}} \leq PoA_{\text{CCE}} \leq \frac{\lambda}{1 - \mu} .$$

For many classes of games, there are choices of λ and μ such that all relations become equalities. These games are referred to as *tight*.

Theorem 5.4. Every congestion game with affine delay functions is $(\frac{5}{3}, \frac{1}{3})$ -smooth. Thus, the PoA is upper bounded by $\frac{5}{2} = 2.5$, even for coarse-correlated equilibria.

We use the following lemma:

Lemma 5.5 (Christodoulou, Koutsoupias, 2005). *For all integers $y, z \in \mathbb{Z}$ we have*

$$y(z+1) \leq \frac{5}{3} \cdot y^2 + \frac{1}{3} \cdot z^2 .$$

Proof. Consider the case $y = 1$. Note that, as z is an integer, we have $(z-1)(z-2) \geq 0$. Therefore, we have

$$z^2 - 3z + 2 = (z-1)(z-2) \geq 0 ,$$

which implies

$$z \leq \frac{2}{3} + \frac{1}{3}z^2 ,$$

and therefore

$$y(z+1) = z+1 \leq \frac{5}{3} + \frac{1}{3}z^2 = \frac{5}{3}y^2 + \frac{1}{3}z^2 .$$

Now consider the case $y > 1$. We now use

$$0 \leq \left(\sqrt{\frac{3}{4}}y - \sqrt{\frac{1}{3}}z \right)^2 = \frac{3}{4}y^2 + \frac{1}{3}z^2 - yz .$$

Using $y \leq \frac{y^2}{2}$, we get

$$y(z+1) = yz + y \leq \frac{3}{4}y^2 + \frac{1}{3}z^2 + \frac{1}{2}y^2 \leq \frac{5}{3}y^2 + \frac{1}{3}z^2 .$$

Finally, for the case $y \leq 0$, observe that the claim is trivial for $y = 0$ or $z \geq -1$, because $y(z+1) \leq 0 \leq \frac{5}{3} \cdot y^2 + \frac{1}{3} \cdot z^2$. In the case that $y < 0$ and $z < -1$, we use that $y(z+1) \leq |y|(|z|+1)$ and apply the bound for positive y and z shown above. \square

Proof of Theorem 5.4. Given two states S and S^* , we have to bound

$$\sum_{i \in \mathcal{N}} c_i(S_i^*, S_{-i}) .$$

We have

$$c_i(S_i^*, S_{-i}) = \sum_{r \in S_i^*} d_r(n_r(S_i^*, S_{-i})) .$$

Furthermore, as all d_r are non-decreasing, we have $d_r(n_r(S_i^*, S_{-i})) \leq d_r(n_r(S) + 1)$. This way, we get

$$\sum_{i \in \mathcal{N}} c_i(S_i^*, S_{-i}) \leq \sum_{i \in \mathcal{N}} \sum_{r \in S_i^*} d_r(n_r(S) + 1) .$$

By exchanging the sums, we have

$$\sum_{i \in \mathcal{N}} \sum_{r \in S_i^*} d_r(n_r(S) + 1) = \sum_{r \in \mathcal{R}} \sum_{i: r \in S_i^*} d_r(n_r(S) + 1) = \sum_{r \in \mathcal{R}} n_r(S^*) d_r(n_r(S) + 1) .$$

To simplify notation, we write n_r for $n_r(S)$ and n_r^* for $n_r(S^*)$. Recall that delays are $d_r(n_r) = a_r n_r + b_r$. In combination, we get

$$\sum_{i \in \mathcal{N}} c_i(S_i^*, S_{-i}) \leq \sum_{r \in \mathcal{R}} (a_r(n_r + 1) + b_r) n_r^* ,$$

Let us consider the term for a fixed $r \in \mathcal{R}$. We have

$$(a_r(n_r + 1) + b_r) n_r^* = a_r(n_r + 1) n_r^* + b_r n_r^* .$$

Lemma 5.5 implies that

$$(n_r + 1)n_r^* \leq \frac{1}{3}n_r^2 + \frac{5}{3}(n_r^*)^2 .$$

Thus, we get

$$\begin{aligned} (a_r(n_r + 1) + b_r)n_r^* &\leq \frac{1}{3}a_r n_r^2 + \frac{5}{3}a_r(n_r^*)^2 + b_r n_r^* \\ &\leq \frac{1}{3}a_r n_r^2 + \frac{1}{3}b_r n_r + \frac{5}{3}a_r(n_r^*)^2 + \frac{5}{3}b_r n_r^* \\ &= \frac{1}{3}(a_r n_r + b_r)n_r + \frac{5}{3}(a_r n_r^* + b_r)n_r^* . \end{aligned}$$

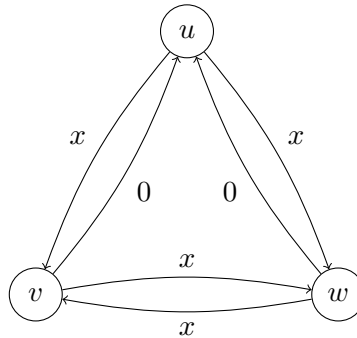
Summing up these inequalities for all resources $r \in \mathcal{R}$, we get

$$\begin{aligned} \sum_{r \in \mathcal{R}} (a_r(n_r + 1) + b_r)n_r^* &\leq \frac{5}{3} \sum_{r \in \mathcal{R}} (a_r n_r^* + b_r)n_r^* + \frac{1}{3} \sum_{r \in \mathcal{R}} (a_r n_r + b_r)n_r \\ &= \frac{5}{3} \cdot \text{cost}(S^*) + \frac{1}{3} \cdot \text{cost}(S) , \end{aligned}$$

which shows $(\frac{5}{3}, \frac{1}{3})$ -smoothness. □

Theorem 5.6. *There are congestion games with affine delay functions whose price of anarchy for pure Nash equilibria is $\frac{5}{2}$.*

Proof sketch. We consider the following (asymmetric) network congestion game. Notation 0 or x on an edge means that $d_r(x) = 0$ or $d_r(x) = x$ for this edge.



There are four players with different source sink pairs. Refer to this table for a socially optimal state of social cost 4 and a pure Nash equilibrium of social cost 10.

player	source	sink	strategy in OPT	cost in OPT	strategy in PNE	cost in PNE
1	u	v	$u \rightarrow v$	1	$u \rightarrow w \rightarrow v$	3
2	u	w	$u \rightarrow w$	1	$u \rightarrow v \rightarrow w$	3
3	v	w	$v \rightarrow w$	1	$v \rightarrow u \rightarrow w$	2
4	w	v	$w \rightarrow v$	1	$w \rightarrow u \rightarrow v$	2

□

Recommended Literature

- Chapters 18 and 19.3 in the AGT book. (PoA and PoS bounds)
- B. Awerbuch, Y. Azar, A. Epstein. The Price of Routing Unsplittable Flow. STOC 2005. (PoA for pure NE in congestion games)
- G. Christodoulou, E. Koutsoupias. The Price of Anarchy of finite Congestion Games. STOC 2005. (PoA for pure NE in congestion games)

- A. Blum, M. Hajiaghayi, K. Ligett, A. Roth. Regret Minimization and the Price of Total Anarchy. STOC 2008 (PoA for No-Regret Sequences)
- T. Roughgarden. Intrinsic Robustness of the Price of Anarchy. STOC 2009. (Smoothness Framework and Unification of Previous Results)