Algorithmic Game Theory, Summer 2015 Lecture 6 (4 pages)

Price of Stability and Strong Nash Equilibria

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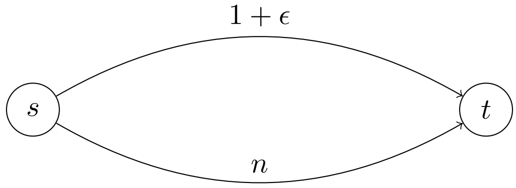
For this lecture, we turn to *fair cost sharing games*, which are congestion games with delays $d_r(x) = c_r/x$ for constant c_r for every resource $r \in \mathcal{R}$. That is, we have a set $\mathcal{N}$ of n players and a set $\mathcal{R}$ of m resources. Player i allocates some resources, i.e., his strategy set is $\Sigma_i \subseteq 2^{\mathcal{R}}$. Each resource $r \in \mathcal{R}$ has fixed cost $c_r \geq 0$. The cost c_r is assigned in equal shares to the players allocating r (if any).

Social cost turns out to be the sum of costs of resources allocated by at least one player:

$$cost(S) = \sum_{i \in \mathcal{N}} c_i(S) = \sum_{i \in \mathcal{N}} \sum_{r \in S_i} d_r(n_r) = \sum_{\substack{r \in \mathcal{R} \\ n_r \geq 1}} n_r \cdot c_r/n_r = \sum_{\substack{r \in \mathcal{R} \\ n_r \geq 1}} c_r \enspace .$$

The price of anarchy for pure Nash equilibria can be as big as the number of players n, even in a symmetric game. For $\epsilon > 0$, consider the example



Edge labels indicate the cost value c_r for this resource. It is a pure Nash equilibrium if all players use the bottom edge, whereas the social optimum would be that all users use the top edge.

Although this is a very stylized example, there are indeed examples of such bad equilibria occurring in reality. A prime example are mediocre technologies, which win against better ones just because they are in the market early and get their share. This way, they are widely supported. Maybe another example are social networks and messaging apps. Many people would prefer not to use, say, Facebook but they cannot switch to an alternative platform unless their friends do.

That said, the price-on-anarchy viewpoint is still a pessimistic one and in today's lecture we will learn shows that when being just a little more positive, we can show much better bounds.

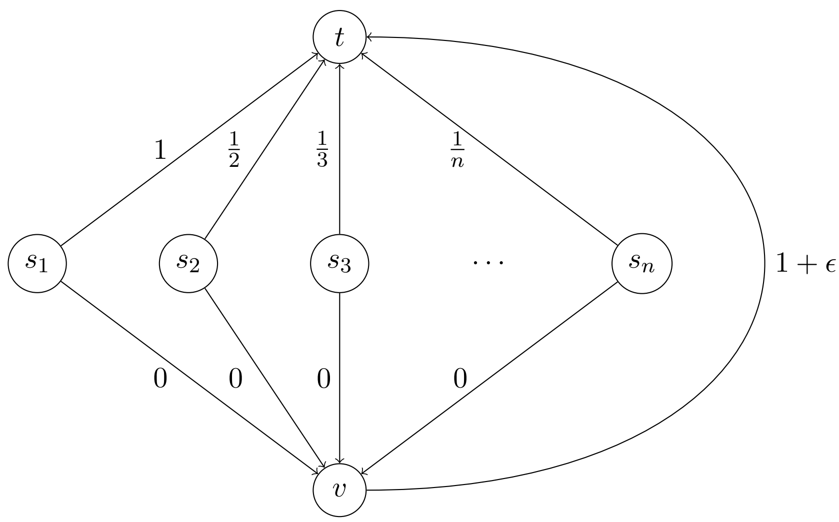
1 Price of Stability

Let us first turn to the *price of stability*. For an equilibrium concept Eq, it is defined as

$$PoS_{\mathsf{Eq}} = \frac{\min_{p \in \mathsf{Eq}} cost(p)}{\min_{s \in S} cost(s)} \enspace .$$

Observation 6.1. *In a symmetric cost sharing game, every social optimum is a pure Nash equilibrium. Therefore, the price of stability for pure Nash equilibria is* 1.

For general, asymmetric games, the social optimum is not necessarily a pure Nash equilibrium. Consider the following game with n players. Each player i has source node s_i and destination node t.



A player two possible strategies: Either take the direct edge or take the detour via v. The social optimum lets all players choose the indirect path, ending up with social cost $1+\epsilon$. This, however, is no Nash equilibrium. Player n would opt out and take the direct edge. Therefore, the only pure Nash equilibrium lets all players choose their direct edge, yielding social cost of $\mathcal{H}_n$. Here, $\mathcal{H}_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ denotes the n-th harmonic number. We have $\mathcal{H}_n = \Theta(\log n)$.

Theorem 6.2. *The Price of Stability for pure Nash equilibria in fair cost sharing games is at most $\mathcal{H}_n$.*

Proof. Rosenthal's potential function for cost sharing delays is

$$\begin{aligned} \Phi(S) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{n_r} c_r / i &= \sum_{\substack{r \in \mathcal{R} \\ n_r \geq 1}} c_r \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n_r}\right) \\ &\leq \sum_{\substack{r \in \mathcal{R} \\ n_r \geq 1}} c_r \cdot \mathcal{H}_n \\ &= \mathit{cost}(S) \cdot \mathcal{H}_n \ . \end{aligned}$$

In $\Phi(S)$ we account for each player allocating resource r a contribution of c_r/i for some $i = 1, \ldots, n_r$, whereas in his cost $c_i(S)$ we account only c_r/n_r. Hence, for every state S of a cost sharing game we have

$$\mathit{cost}(S) \leq \Phi(S) \leq \mathit{cost}(S) \cdot \mathcal{H}_n \ .$$

Now suppose we start at the optimum state S^* and iteratively perform improvement steps for single players. This eventually leads to a pure Nash equilibrium. Every such move decreases the potential function. For the resulting Nash equilibrium S we thus have $\Phi(S) \leq \Phi(S^*)$ and

$$\mathit{cost}(S) \leq \Phi(S) \leq \Phi(S^*) \leq \mathit{cost}(S^*) \cdot \mathcal{H}_n \ .$$

This proves that there is a pure Nash equilibrium that is only a factor of $\mathcal{H}_n$ more costly than S^*. □

2 Strong Nash Equilibria

Let us consider again the example that the price of anarchy can be as high as n. There are two pure Nash equilibria, namely all players taking either the cheap or the expensive edge. Observe that the bad equilibrium in which all players take the expensive edge is actually brittle. If any single player convinced one of his friends to take the cheap edge, the whole equilibrium would fall apart. Indeed this is true for all cost sharing games, as we will show next.

First, we will introduce another solution concept called *strong Nash equilibrium.*

Definition 6.3. *Let s be a state of a cost-minimization game. Consider a subset of players $A \subseteq \mathcal{N}$ (coalition). The strategy vector s' is a* beneficial deviation *for A if*

$$\begin{aligned} & c_i(s'_A, s_{-A}) \leq c_i(s) && \text{for all } i \in A \\ \text{and} \quad & c_i(s'_A, s_{-A}) < c_i(s) && \text{for at least one } i \in A \ . \end{aligned}$$

The state s is called a strong Nash equilibrium *if there is no coalition with a beneficial deviation.*

Every strong Nash equilibrium is also a pure Nash equilibrium because unilateral deviations are covered by coalition size 1. In our introductory example, we had two pure Nash equilibria but only one strong Nash equilibrium, namely the one where all players choose the cheaper edge. We will now consider the price of anarchy for strong Nash equilibria. Remember that generally, for an equilibrium concept Eq, it is defined as

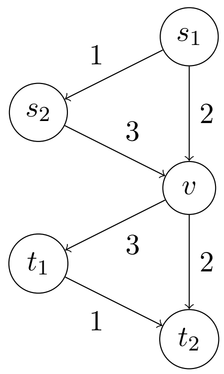
$$PoA_{\mathsf{Eq}} = \frac{\max_{p \in \mathsf{Eq}} cost(p)}{\min_{s \in S} cost(s)} \ .$$

So, given the respective equilibria exist, we have

$$PoS_{\mathsf{PNE}} \leq PoS_{\mathsf{SNE}} \leq PoA_{\mathsf{SNE}} \leq PoS_{\mathsf{PNE}} \ .$$

Observation 6.4. *A state s of a symmetric cost sharing game is a strong Nash equilibrium if and only if it is socially optimal. Therefore, the price of anarchy for strong Nash equilibria is 1.*

An asymmetric cost-sharing game does not always admit a strong Nash equilibrium, which you can verify in this example.



Even if there is a strong Nash equilibrium, the price of anarchy can be as big as $\mathcal{H}_n$. Just consider the same example that showed us that the price of stability for pure Nash equilibria is at least H_n. The only pure Nash equilibrium in this case is also a strong Nash equilibrium. However, we can show an upper bound of $\mathcal{H}_n$ on the social cost a strong Nash equilibrium. So, if there is a strong Nash equilibrium, this bounds the price of anarchy.

Theorem 6.5. *The price of anarchy for strong Nash equilibria in fair cost sharing games is at most $\mathcal{H}_n$.*

Proof. Let S be a strong Nash equilibrium, S^* be a socially optimal state.

First, we consider the coalition that consists of all players. Letting all players deviate to S^* is not beneficial. Therefore, there has to be one player i for which $c_i(S) \leq c_i(S^*)$. Without loss of generality, let this player be n.

Next, we consider the coalition of all players except n. Again, it is not beneficial if these players deviate to S^*. So, again, there has to be a player i for which $c_i(S) \leq c_i(S^*_{-n}, S_n)$. Let this player be $n-1$.

Following the argument, after renaming players, we get strategy profiles S^t for $t \in \{1, 2, \ldots, n\}$ such that

$$S_i^t = \begin{cases} S_i^* & \text{for } i \leq t \\ S_i & \text{for } i > t \end{cases}$$

and $c_t(S) \leq c_t(S^t)$.

For $r \in \mathcal{R}$, define $k_r^t = |\{i \leq t \mid r \in S_i^*\}|$. We now have

$$c_t(S^t) = \sum_{r \in S_t^t} \frac{c_r}{n_r(S^t)} \leq \sum_{r \in S_t^t} \frac{c_r}{k_r^t} \ .$$

This gives us

$$\sum_{i \in \mathcal{N}} c_i(S) \leq \sum_{t=1}^{n} c_t(S^t) \leq \sum_{t=1}^{n} \sum_{r \in S_t^t} \frac{c_r}{k_r^t} = \sum_{r \in \mathcal{R}} c_r \sum_{t : r \in S_t^*} \frac{1}{k_r^t} \ .$$

Now observe that

$$\sum_{t : r \in S_t^*} \frac{1}{k_r^t} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n_r(S^*)} \ .$$

Therefore,

$$\sum_{r \in \mathcal{R}} c_r \sum_{t : r \in S_t^*} \frac{1}{k_r^t} = \Phi(S^*) \ .$$

Consequently,

$$cost(S) = \sum_{i \in \mathcal{N}} c_i(S) \leq \Phi(S^*) \leq \mathcal{H}_n cost(S^*) \ .$$

□

Recommended Literature

- Chapter 19.3 in the AGT book. (PoS bound)
- Tim Roughgarden's lecture notes `http://theory.stanford.edu/~tim/f13/l/l15.pdf` and lecture video `https://youtu.be/VjCKNl-GENI`
- A. Epstein, M. Feldman, and Y. Mansour. Strong equilibrium in cost sharing connection games. Games and Economic Behavior, 67(1):5168, 2009. (bound for strong equilibria)