Motivation
We consider basic operations on polynomials such as root isolation and gcd computation.

GCD Computation
Let \( f \) be a polynomial with rational coefficients. Consider the following recursion: We initially start with \( f_0 := f \) and \( f_1 := f' \). For \( i \geq 0 \) let \( d_i := \deg f_i - \deg f_{i+1} \) and consider
\[
f_{i+2} := \lambda f_i + x^{d_i} f_{i+1}
\]
with rational \( \lambda \) such that \( f_{i+2} \) has lower degree then \( f_i \). Show:

- Let \( g \) be a rational polynomial that divides \( f \) and \( f' \), that is, there exists rational polynomials \( g_1 \) and \( g_2 \) with \( f = g_1 \cdot g \) and \( f' = g_2 \cdot g \). Then \( g \) divides each \( f_i \).
- Let \( i_0 \) be the first index \( i \) where \( f_i = 0 \). Then \( f_{i_0-1} \) divides \( f \) and \( f' \) and there exists no polynomial of larger degree with the same property. It follows that \( f_{i_0-1} = \gcd(f, f') \).
- If \( f \) is a rational polynomial with distinct complex roots \( \xi_1, \ldots, \xi_m \) then
\[
f^* := (x - \xi_1) \cdots (x - \xi_m)
\]
is rational as well and a scalar multiple of \( f / \gcd(f, f') \).
- \( \deg \gcd(f, f') = \deg f - m \) with \( m \) as above.

Real Root Isolation
Given a polynomial \( f = \sum_{i=0}^{n} a_i x^i \) with real coefficients we aim for a set of disjoint intervals \( I_1, \ldots, I_m \) such that their union \( \bigcup_{k=1}^{n} I_k \) contains all real roots of \( f \) and each \( I_k \) contains exactly one real root.

- Show that the modulus \( |\xi| \) of each root \( \xi \) of \( f \) is bounded by
\[
B := 1 + \max_i \frac{|a_i|}{|a_n|}.
\]
(Hint: Each root \( \xi \) of \( f \) fulfills the inequality \( |a_n||\xi|^n \leq \sum_{i=0}^{n} |a_i||\xi|^n \).)
Let $I = (a, b)$ be an interval with midpoint $m = \frac{a+b}{2}$ and $g$ a polynomial of degree $N$ with Taylor expansion

$$g(m + x) = \sum_{k=0}^{N} \frac{g^{(k)}(m)}{k!} x^k$$

at $m$. We consider the test

$$T(g, I) : |g(m)| > \sum_{k=1}^{N} \frac{|g^{(k)}(m)|}{k!} \left( \frac{b-a}{2} \right)^k.$$ 

Show that $I$ contains no root of $f$ if $T(f, I)$ succeeds!

Show: If $T(f', I)$ succeeds then $f$ is monotone on $I$. How can you use this test to show that an interval $I$ is isolating?

Formulate an algorithm to isolate all real roots of $f$ and show exactness and termination.

(Hint: For the root isolation consider $f^* := f / \gcd(f, f')$ and use the fact that $(f^*)'(\xi) \neq 0$ at all roots $\xi$ of $f^*$.)

Have fun with the solution!

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