Bitstream Descartes for multiple roots

We “extend” the Bitstream Descartes to polynomials with multiple roots: Let \( f \) be a polynomial with real coefficients and \( k \) denote the maximal multiplicity of a root of \( f \).

- Show that there exists a \( w_0 > 0 \) such that for all intervals of size \( w(I) < w_0 \) it holds that \( \text{var}(f, I) \leq k \). Give a bound on \( w_0 \) in terms of the separation of \( f \) (= minimal distance between two distinct roots).

- We call a polynomial generic iff it has a root of multiplicity \( k + 1 = \deg \gcd(f, f') + 1 \). Show that, for \( k > 0 \), each generic polynomial has exactly one multiple root \( z \). Furthermore, show that \( z \) is real and all remaining roots of \( f \) are simple.

- Now, \( f \) is a polynomial with bitstream coefficients. Furthermore, we assume that \( k := \deg \gcd(f, f') \) is known and that we can ask for an arbitrary good approximation \( \tilde{f}^* \) of the square-free part \( f^* := f / \gcd(f, f') \) of \( f \). Formulate an algorithm to
  1. determine isolating intervals \( I_1, \ldots, I_m \) for the real roots of \( f \).
  2. refine the intervals \( I_j \) to any specified size.
  3. determine whether \( f \) is generic or not.
  4. determine which of the intervals \( I_j \) contains the unique multiple root of \( f \).

Topology of a Planar Curve

Determine the topology of the planar curve \( C := V_{\mathbb{R}}(x^3 - 2xy + 2y^2 + x^2) \), that is, compute an isocomplex for \( C \).

Why is this argumentation wrong?

Let \( C \) be a planar algebraic curve. We are interested in a shearing of \( C \) such that the transformed curve \( C' \) is in general position, that is, no two critical points are co-vertical. We want to show that all but finitely many shearing directions induce a curve \( C' \) in general position:

It suffices to find a direction \( \phi \in [0, 2\pi) \) such that each line pointing toward the direction \( \phi \) does not pass two or more critical points of \( C \). There exists only finitely many critical points \( p_1, \ldots, p_m \) of \( C \). Let \( \phi_i, i = 1, \ldots, \binom{m}{2} \) denote the directions defined by each pair of
critical points, then, each direction $\phi \neq \phi_i$ defines a shearing which induces a curve in general position. This shows our claim.

Have fun with the solution!