Optimization II	**Winter 2009/10**

Lecture 9 — January 4

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Let us recall a few basic concept form polyhedral theory. A polyhedron is a set $P = \{x \in R^m \mid xA \leq b\}$ defined by a matrix $A \in R^{n \times m}$ and a vector $b \in R^m$. For a given $x \in P$, let A' and b' be the set of *tight constraints* at x. We say x is a *basic solution* if it is the unique solution to the sytem $A'x = b'$, and the matrix A' is called *basic matrix*[1] of x. The set of vertices or extreme points of P is precisely the set of basic feasible solutions of P. A polyhedron is integral if *all* its vertices are integral.

The main objective of this lecture is to identify matrix properties that make the polyhedra defined by these matrices integral. These notes are based mostly on a survey paper of Conforti, Cornuéjols, and Vušković.

9.1 Balanced matrices and integral polyhedra

Definition 9.1. *A square matrix A is unimodular if* $\det A \in \{0, -1, 1\}$.

By Cramer's rule, the inverse of an integral unimodular matrix is also integral. This type of matrices are very useful because if the basic matrix associated with a given feasible x is unimodular then x will be integral as long as the vector b is integral.

Definition 9.2. *A matrix A is totally unimodular (TU) if every square submatrix of A is unimodular.*

As a concequence, TU matrices are $\{0, -1, 1\}$ matrices: If A has an entry $a \notin \{0, -1, 1\}$ then the 1×1 matrix containing just a has determinant a, which violates unimodularity.

Theorem 9.3. *Let $A \in \{0, 1, -1\}^{n \times m}$ be a TU matrix and b be an arbitrary integral vector. Then the polyhedron $\{x \in R^m \mid Ax \leq b, x \geq 0\}$ is integral.*

While total unimodularity is a nice property to have, some times a weaker notion is enough to prove integrality of *certain* polyhedra. A hole a $\{0, 1\}$ matrix with two ones per column and two ones per row such that no submatrix has this property.

Definition 9.4. *A matrix $\{0, 1\}^{n \times m}$ is balanced (BA) if it is does not contain an odd hole as a submatrix.*

Theorem 9.5. *Let $A \in \{0, 1\}^{n \times m}$ be BA. The following polyhedra are integral:*

$$\{x \in R^m \mid Ax \leq 1, x \geq 0\}, \{x \in R^m \mid Ax \geq 1, x \geq 0\}, \text{ and } \{x \in R^m \mid Ax = 1, x \geq 0\}.$$

As we will see TU matrices are a sub-class of BA. We finish this section with another interesting sub-class of BA matrices.

[1] To be more precise, a submatrix thereof with full rank.

Definition 9.6. *A matrix $\{0,1\}^{n\times m}$ is totally balanced (TB) if it is does not contain a hole of length 3 or more as a submatrix.*

TB matrices not only have integral polyhedra, but there also exist a simple primal-dual algorithm for solving the linear optimization problem associated with these polyhedra. In addition, they have a surprising alternative characterization, which makes them easy to identify.

9.2 Alternative characterizations

A $\{0,1\}$ matrix is *bicolorable* if its columns can be colored with RED and BLUE so that every no row containing at least two 1s is monochromatic.

Theorem 9.7. *A $\{0,1\}$ matrix A is BA if and only if every submatrix of A is bicolorable.*

Theorem 9.8. *A $\{0,1\}$ matrix A is TB if and only if its rows and columns can be permuted so that it does not contain*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

as a submatrix.

A $\{0,-1,1\}$ matrix has an *equitable bicoloring* if its columns can colored with RED and BLUE so that in every, the sum of the blue entries differs from the sum of the red entries by at most 1.

Theorem 9.9. *A $\{0,-1,1\}$ matrix A is TU if and only if every submatrix of A has an equitable bicoloring.*

With this new characterization in hand it is easy to see that every TU is also BA. Notice, however, that the opposite happens in he case of TB matrices!

9.3 Examples

Lemma 9.10. *Let G be an undirected bipartite graph with n vertices and m edges, and let $A \in \{0,1\}^{n\times m}$ be the vertex-edge indicence matrix of G. Then the matrix A is TU.*

Corollary 9.11 (König's Theorem). *The size of the largest matching in a bipartite graph equals the size of the smallest vertex cover.*

Let T be a tree and $\{(s_i,t_i)\}_{i=1}^k$ be a collection of source-sink pairs. Each pair (s_i,t_i) define a unique path p_i in T. A multicut is a set of edges whose removal disconnects every path. We say the instance has ascending path if the tree T can be rooted so that s_i is a proper ancestor of t_i, or vise-versa, for all $i = 1,\ldots,k$.

Let A be a the incidence matrix between the set of paths $\{p_i\}_{i=1}^k$ and the edges of T and; we call A the *multicut matrix* of the given multicut instance.

Lemma 9.12. *Let A be the multicut matrix of an instance with ascending paths. Then the matrix A is TB and TU.*