Expander Graphs in Computer Science **WS 2010/2011**

Lecture 2: Algebraic Definitions of Expanders

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1 Graph Spectrum

Let G be a d-regular undirected graph with n vertices, and the adjacency matrix of G is represented by

$$A := (a_{i,j})_{n\times n}$$

where $a_{i,j}$ is the number of edges between vertex i and vertex j. In addition, we define the following two matrices:

- The Laplacian matrix of G is defined by $L := dI - M$.
- The normalized adjacency matrix of G is $M := A/d$.

Let us focus on the normalized adjacency matrix in this course.

Lemma 2.1 *Let G be a d-regular undirected graph with n vertices and M is the normalized adjacency matrix of G. We have*

- *M is real-symmetric.*
- *The sum of the elements in each row/column is 1.*

Definition 2.2 *Given a matrix A, a vector x is defined to be an eigenvector of A if and only if there is a λ such that $Ax = \lambda x$. In this situation, λ is called an eigenvalue of A.*

When considering graphs as matrices, we study the properties of the adjacency matrix of graphs and use algebraic methods to study graphs. This branch in graph theory is called algebraic graph theory.

Definition 2.3 (graph spectrum) *Given a d-regular graph G with n vertices, the normalized adjacency matrix of G is defined by M. Then, M has n real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. These eigenvalues compose the spectrum of G.*

Lemma 2.4 *If G is regular, then $\lambda_1 = 1$ and $|\lambda_i| \leq 1$ for $i = 2, \cdots, n$.*

Lemma 2.5 *(1) If G is connected then G is bipartite iff -1 is also an eigenvalue of M. (2) G is connected iff the eigenvalue 1 has multiplicity 1.*

For a graph G with adjacency matrix A, it is easy to see that $\mathrm{tr}(A^k)$ is the number of all walks of length k in G that start and end in the same vertex.

Lemma 2.6 *For any graph G with m edges, the number of triangles in G is bounded by $\frac{\sqrt{2}}{3} \cdot m^{3/2}$.*

Proof: Let A be the adjacency matrix of G with eigenvalues $\lambda_1, \cdots, \lambda_n$. Thus the number of triangles is bounded by $\mathrm{tr}(A^3)/6 = \left(\sum_{i=1}^n \lambda_i^3\right)/6$. Since for any $k \geq 3$, there holds that

$$\left(\sum_{i=1}^n \lambda_i^k\right)^{1/k} \leq \left(\sum_{i=1}^n |\lambda_i|^k\right)^{1/k} \leq \left(\sum_{i=1}^n |\lambda_i|^2\right)^{1/2} = (2 \cdot m)^{1/2}.$$

Therefore the number of triangles is at most $\frac{\sqrt{2}}{3} \cdot m^{3/2}$. ■

2 Spectral Expansion v.s. Combinatorial Expansion

Definition 2.7 (spectral expansion) *G has spectral expansion λ if* $\max\{|\lambda_2|, |\lambda_n|\} \leq \lambda$.

Small spectral expansion implies better vertex expansion.

Theorem 2.8 (spectral expansion⇒ vertex expansion) *If G has spectral expansion λ, then for all α, G has vertex expansion $\left(\alpha n, \frac{1}{(1-\alpha)\lambda^2+\alpha}\right)$.*

Before showing the proof, we introduce some notations at first. For any probability distribution π, the support of π is defined by $\mathrm{support}(\pi) = \{x : \pi_x > 0\}$.

Definition 2.9 *Given a probability distribution π, the collision probability of π is defined to be the probability that two independent samples from π are equal, i. e.* $\mathrm{CP}(\pi) = \sum_x \pi_x^2$.

Lemma 2.10 *For every probability distribution $\pi \in [0,1]^N$, we have*

1. $\mathrm{CP}(\pi) = ||\pi||^2 = ||\pi - u||^2 + 1/N$.
2. $\mathrm{CP}(\pi) \geq 1/|\mathrm{support}(\pi)|$.

Proof: (1) We write π as $\pi = u + (\pi - u)$ where $u \perp (\pi - u)$. Thus

$$\mathrm{CP}(\pi) = ||\pi||^2 = ||\pi - u||^2 + ||u||^2 = ||\pi - u||^2 + 1/N.$$

(2) By Cauchy-Schwarz inequality, we get

$$1 = \sum_{x \in \mathrm{support}(\pi)} \pi_x \leq \sqrt{|\mathrm{support}(\pi)|} \cdot \sqrt{\sum_x \pi_x^2}$$

and $\mathrm{CP}(\pi) = \sum_x \pi_x^2 \geq 1/|\mathrm{support}(\pi)|$. ■

Proof:[of Theorem 2.8] Let $|S| \leq \alpha N$. Choose a probability distribution π that is uniform on S and 0 on the $\overline{S}$. Then $\mathrm{CP}(\pi) = 1/|S|$ and $\mathrm{CP}(M\pi) \geq 1/|\mathrm{support}(M\pi)| = 1/|\Gamma(S)|$. Therefore $1/|\Gamma(S)| - 1/N \leq \lambda^2(1/|S| - 1/N)$. Combing the formula above and $|S| \leq \alpha N$, we get

$$|\Gamma(S)| \geq \frac{|S|}{(1-\alpha)\lambda^2 + \alpha}.$$

■

Theorem 2.11 (vertex expansion $\Rightarrow$ spectral expansion) *For every $\delta > 0$ and $d > 0$, there is $\gamma > 0$ such that if G is a d-regular $(1+\delta)$-expander, then it is also $(1-\gamma)$ spectral expander. Specifically, we can take $\gamma = \Omega(\delta^2/d)$.*

When talking about expanders, we often mean a family of d-regular graphs satisfying one of the following two equivalent properties:

- Every graph in the family has spectral expansion λ.
- Every graph in the family is a $(1+\delta)$-expander for some constant δ.

Theorem 2.12 *Any infinite family of d-regular graphs has spectral expansion (as $N \to \infty$) at least $2\sqrt{d-1}/d - o(1)$.*

Definition 2.13 (Ramanujan graphs) *A family of d-regular graphs with spectral expansion at most $2\sqrt{d-1}/d$ is called Ramanujan graphs.*

3 Expander Mixing Lemma

Lemma 2.14 (Expander Mixing Lemma) *Let $G = (V, E)$ be a d-regular n-vertex graph with spectral expansion λ. Then $\forall S, T \subseteq V$, we have*

$$\left| |E(S,T)| - \frac{d|S| \cdot |T|}{n} \right| \leq \lambda d \sqrt{|S||T|}.$$

Let us consider the two terms in the left side: the size of $E(S,T)$ is the number of edges between two sets, and $\frac{d|S|\cdot|T|}{n}$ is the expected number of edges between S and T in a random graph with edge density d/n. So small λ implies that G is "more" random.

Proof: Let $\mathbf{1}_S, \mathbf{1}_T$ be the characteristic vectors of S and T. Expand these vectors in the orthonormal basis of eigenvectors $v_1, \cdots, v_n$, i. e. $\mathbf{1}_S = \sum_i \alpha_i v_i$, and $\mathbf{1}_T = \sum_i \beta_i v_i$. Then

$$|E(S,T)| = \mathbf{1}_S \cdot A \cdot \mathbf{1}_T = \left(\sum_i \alpha_i v_i \right) A \left(\sum_i \beta_i v_i \right) = \sum_i \lambda_i \alpha_i \beta_i,$$

where λ_is are eigenvalues of A. Since $\alpha_1 = \langle \mathbf{1}_S, \frac{1}{\sqrt{n}} \rangle = \frac{|S|}{\sqrt{n}}, \beta_1 = \frac{|T|}{\sqrt{n}}$ and $\lambda_1 = d$, then

$$|E(S,T)| = d \cdot \frac{|S| \cdot |T|}{n} + \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i.$$

Thus

$$\left| E(S,T) - \frac{d|S||T|}{n} \right| \leq \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i \leq \lambda \cdot d \cdot \sum_{i=2}^{n} |\alpha_i \beta_i|$$

By Cauchy-Schwartz inequality, we have

$$\left| E(S,T) - \frac{d|S||T|}{n} \right| \leq \lambda ||\mathbf{1}_S|| \cdot ||\mathbf{1}_T|| = \lambda \cdot d \cdot \sqrt{|S| \cdot |T|}.$$

■

Lemma 2.15 (Converse of the Expander Mixing Lemma) *Let G be a d-regular graph and suppose that*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \rho\sqrt{|S||T|}$$

holds for every two disjoint sets S, T and for some positive ρ. Then $\lambda \leq O\left(\rho/d \cdot (1 + \log(d/\rho))\right)$.

In the following, we use a three-tuple (n, d, λ) to represent an n-vertex d-regular graph with spectral expansion λ.

Corollary 2.16 *The size of the independent set for any (n, d, λ)-graph is at most λn.*

Proof: Let $T = S$. By Expander Mixing Lemma, we get $|S| \leq \lambda n$. ■

Corollary 2.17 *For any (n, d, λ)-graph G, the chromatic number $\chi(G) \geq 1/\lambda$.*

Proof: Let $c : V \to \{1, \cdots, k\}$ be a coloring of G. Then for every $1 \leq i \leq k$, $c^{-1}(i)$ is an independent set. Since the size of every independent set is at most λn, so the chromatic number is at least $1/\lambda$. ■