Expander Graphs in Computer Science **WS 2010/2011**

Lecture 3: Random Walks on Expanders

Lecturer: He Sun

Consider an undirected graph G. A random walk of length ℓ staring at vertex u is a sequence of vertices $u = v_0, \cdots, v_\ell$, where each v_i is chosen uniformly from the neighbors of v_{i-1}, for $i > 0$.

Traditionally, random walks were studied in infinite graphs. One of the basic results is as follows:

Theorem 3.1 (Pólya, 1921) *Consider a random walk on an infinite D-dimensional grid. If $D = 2$, then with probability 1, the walk returns to the starting point an infinite number of times. If $D > 2$, then with probability 1, the walk returns to the starting point only a finite number of times.*

In this lecture, we assume that G is d-regular. Let π_0 be the initial distribution of vertices in G and π_i be the probability distribution of v_i for a random walk beginning at vertex v_0. Then we observe the following facts.

- $\pi_1 = A\pi_0$.
- $\pi_{i+1} = A\pi_i = A(A\pi_{i-1}) = A^{i+1}\pi_0$.

Definition 3.2 *A distribution π is called stationary if and only if $\pi = A\pi$.*

For considering the convergence of $\{\pi_i\}$, we get the following result:

Lemma 3.3 *For every finite connected un-bipartite graph G, the distribution π_i converges to a limit and stationary distribution. Moreover, if G is regular, then this distribution is the uniform distribution $\mathbf{u}$ on V.*

This lemma implies that, for any initial distribution π_0, if we take the random walk on an expander G, then after a finite number of steps, the probability that the random walk hits every vertex is uniform. However, for practical interests, we ask how fast π_i converges to $\mathbf{u}$.

Definition 3.4 *For any vector $\mathbf{x}$, define*

$$\|\mathbf{x}\|_p = \left(\sum_i |x_i|^p\right)^{1/p}.$$

In particular, let $\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$.

Definition 3.5 (Mixing Time) *The mixing time of a graph G with n vertices is the minimum ℓ such that for any staring distribution π,*

$$\left\|A^\ell \pi - \mathbf{u}\right\|_\infty < \frac{1}{2n}.$$

A key property of random walks on an expander is that it converges rapidly to limit distributions.

Theorem 3.6 *If G is a connected, d-regular, non-bipartite graph on n vertices, then $\lambda < 1$ and G has mixing time $O\left(\frac{\log n}{1-\lambda}\right)$.*

Proof: Let $\mathbf{u} = v_1, \cdots, v_n$ be the orthonormal eigenvalues of A. Then for any distribution π on the vertices of G, we can write $\pi = \sum_{i=1}^{n} \pi_i$, where π_i is a constant multiple of v_i. Therefore

$$\begin{aligned}
||A\pi - \mathbf{u}||^2 &= ||A\mathbf{u} + A\pi_2 + \cdots + A\pi_n - \mathbf{u}||^2 \\
&= ||\lambda_2\pi_2 + \lambda_3\pi_3 \cdots + \lambda_n\pi_n||^2 \\
&= \lambda_2^2||\pi_2||^2 + \cdots + \lambda_n^2||\pi_n||^2 \\
&\leq \lambda^2\big(||\pi_2||^2 + \cdots + ||\pi_n||^2\big) \\
&= \lambda^2\big(||\pi_2 + \cdots + \pi_n||^2\big) \\
&= \lambda^2||\pi - \mathbf{u}||^2
\end{aligned}$$

which implies that $\|A^\ell\pi - \mathbf{u}\| \leq \lambda^\ell\|\pi - \mathbf{u}\|$. Thus

$$\begin{aligned}
\|A^\ell\pi - \mathbf{u}\| &\leq \lambda^\ell\|\pi - \mathbf{u}\| \\
&\leq \lambda^\ell\|\pi\| \\
&\leq \lambda^\ell\|\pi\|_1 \\
&= \lambda^\ell
\end{aligned}$$

and $\|A^\ell\pi - \mathbf{u}\|_\infty < \frac{1}{2n}$ when $\ell = O\left(\frac{\log n}{\log\frac{1}{\lambda}}\right) = O\left(\frac{\log n}{1-\lambda}\right)$. ■

The above theorem also indicates that small λ implies fast convergence of random walks.

Another important property of random walks is offered by the entropy of associated probability distribution. For any distribution π, the Shannon entropy of π is defined by

$$\mathbf{H}(\pi) = -\sum_{i=1}^{n} \pi_i \log \pi_i.$$

Theorem 3.7 *Let A be the normalized adjacency matrix of an expander G. Then for any distribution π on vertices of G, we have $\mathbf{H}(A\pi) \geq \mathbf{H}(\pi)$.*