Lecture 6: Undirected Connectivity in Log-Space

Lecturer: He Sun

We consider the undirected connectivity problem. Given an undirected graph G represented by adjacency matrix and two vertices s and t, the *undirected connectivity problem* is to decide whether there is a path from s to t. Formally we define the language USTCON.

Definition 6.1 *USTCON is defined as a set of triples (G, u, v) where $G = (V, E)$ is an undirected graph, u, v are two vertices in G so that there is a path from u to v in G.*

This problem has received a lot of attention in the past few decades and the complexity of USTCON has been well studied. The first randomized log-space algorithm for USTCON was shown in 1979 by Aleliunas, Karp, Lipton, Lovász and Rackoff. In 1970, Savitch demonstrated a simulation of a non-deterministic space S machine by a deterministic space S^2 machine. Thus $\mathsf{USTCON} \in \mathrm{SPACE}\left(\log^2 n\right)$. Nisan, Szemerdi and Wigderson in 1989 showed that $\mathsf{USTCON} \in \mathrm{SPACE}\big(\log^{3/2} n\big)$. Armoni, Ta-Shma, Wigderson and Zhou in 2000 proved that $\mathsf{USTCON} \in \mathrm{SPACE}\big(\log^{4/3} n\big)$. In 2005, Reingold used expander graphs to show that USTCON is in $\mathbf{L}$ and $\mathbf{SL}$ collapses to $\mathbf{L}$.

In this lecture we present Reingold's algorithm for the USTCON problem.

1 Algorithm

Let G be an input graph. At a high-level overview, the algorithm reduces G to an expander G_ℓ such that

- $|V[G_\ell]| = \mathrm{poly}(|V[G]|)$.
- G_ℓ is regular and the degree of G_ℓ is constant.
- For any two vertices u and v in G, u and v are connected if and only if the vertices in G_ℓ that correspond to u and v are also connected.
- Each connected component of G_ℓ is an expander. (The spectral expansion is at most $1/2$.)

Therefore for any two vertices u and v in G, u and v are connected if and only if there is a path of length $O\left(\log |V[G_\ell]|\right) = O(\log |V[G]|)$ to connect the vertices in G_ℓ that correspond to u and v.

In the preprocessing step, we would like to transform the input graph G into a D^{16}-regular graph G_1. Now let G_1 be a D^{16}-regular graph on $[N]$ and H is a $\left(D^{16}, D, 1/2\right)$-graph. The existence of such graphs is proven by probabilistic methods and for a constant D, such a graph can be obtained by exhaustive search. Moreover, we can express H by the rotation map in constant time.

Let ℓ be the smallest integer such that $\left(1 - \frac{1}{DN^2}\right)^{2^\ell} \leq 1/2$. The algorithm is as follows.

- For i=1 to $\ell = \mathcal{O}(\log |V[G_0]|)$ do $G_{i+1} = (G_i \circled{z} H)^8$
- Check if s and t are connected in G_ℓ by enumerating over all $O(\log N)$ paths originating at s.

Note that each G_i is a D^{16}-regular graph over $[N] \times \left(\left[D^{16}\right]\right)^i$. Since D is constant and $\ell = O(\log N)$, G_ℓ has poly(N) vertices.

2 Analysis

The working space of the algorithm comes from two aspects: The space for calculating G_i iteratively and the space for deciding the connectivity between s and t in G_ℓ.

Now assume that the input graph G is connected and we prove that G_ℓ is an expander.

Lemma 6.2 *For every D-regular, connected, non-bipartite graph G on $[N]$ it holds that $\lambda(G) \leq 1 - 1/DN^2$.*

Theorem 6.3 *If $\lambda(H) \leq 1/2$, then $1 - \lambda(G \circled{z} H) \geq 1/3 \cdot (1 - \lambda(G))$.*

Theorem 6.4 *For $i = 1, \cdots, \ell$, we have $\lambda(G_i) \leq \max\left\{\lambda^2(G_{i-1}), 1/2\right\}$.*

Proof: Because $G_i = (G_{i-1} \circled{z} H)^8$, by Theorem 6.3 we have

$$\lambda(G_i) = \lambda^8(G_{i-1} \circled{z} H) \leq \left(1 - \frac{1}{3} \cdot (1 - \lambda(G_{i-1}))\right)^8.$$

We consider the following two cases.

(1) $\lambda(G_i) \leq 1/2$. Then

$$\lambda(G_i) = \lambda^8(G_{i-1} \circled{z} H) \leq \left(1 - \frac{1}{3} \cdot \left(1 - \frac{1}{2}\right)\right)^8 \leq \left(\frac{5}{6}\right)^8 \leq \frac{1}{2}.$$

(2) $\lambda(G_i) > 1/2$. Because for any $x \in [1/2, 1]$ it holds

$$\left(1 - \frac{1}{3} \cdot (1 - x)\right)^4 \leq x,$$

we have

$$\lambda(G_i) = \lambda^8(G_{i-1} \circled{z} H) \leq \left(1 - \frac{1}{3} \cdot (1 - \lambda(G_{i-1}))\right)^8 \leq \lambda^2(G_{i-1}).$$

Therefore for any $i \in \{1, \cdots, \ell\}$, $\lambda(G_i) \leq \max\left\{\lambda^2(G_{i-1}), 1/2\right\}$. ■

Corollary 6.5 *The spectral expansion of each connected component of G_ℓ is at most $1/2$.*

Proof: By Lemma 6.2 and Theorem 6.4. ■

Lemma 6.6 *For every constant D, the transformation of G_i can be computed in space $O(\log N)$ on inputs G and H, where G is a D^{16}-regular graphs on $[N]$ and H is a D-regular graph on $\left[D^{16}\right]$.*

Therefore we can implement the algorithm in space $O(\log n)$.

Theorem 6.7 *USTCON $\in$ **L**.*

Corollary 6.8 **SL** $=$ **L**.

Proof: Because USTCON is an **SL**-complete problem, thus USTCON $\in$ **L** implies **SL** $=$ **L**. ■

Appendix

Definition 6.9 *The complexity class* $\mathbf{L}$ *consists of the language decidable within deterministic logarithmic space.*

Definition 6.10 $\mathbf{SL}$ *is the class of problems solvable by a nondeterministic Turing machine in logarithmic space, such that:*

1. *If the answer is 'yes', one or more computation paths accept.*
2. *If the answer is 'no', all paths reject.*
3. *If the machine can make a nondeterministic transition from configuration A to configuration B, then it can also transition from B to A. (This is what 'symmetric' means.)*

The current view of log-space complexity classes is

$$\mathbf{L} \subseteq \mathbf{SL} \subseteq \mathbf{RL} \subseteq \mathbf{NL} \subseteq \mathbf{L}^2.$$

Open Problem 1 $\mathbf{RL} = \mathbf{L}$*?*